INFORMATION, NO-ARBITRAGE AND COMPLETENESS FOR ASSET PRICE MODELS WITH A CHANGE POINT

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Abstract. We consider a general class of continuous asset price models where the drift and the volatility functions, as well as the driving Brownian motions, change at a random time τ. Under minimal assumptions on the random time and on the driving Brownian motions, we study the behavior of the model in all the filtrations which naturally arise in this setting, establishing martingale representation results and characterizing the validity of the NA1 and NFLVR no-arbitrage conditions.

1. Introduction

The behavior of financial asset prices is often subject to certain random events that result in abrupt changes in their dynamics. The random time when such an event occurs is called a change point. For example, a sudden adjustment in the interest rates, a default of a major financial institution, a natural catastrophe, or the release of some political news could all have an impact on the asset price. Although these events are not caused by the price evolution of the individual asset, their occurrence may change the asset price dynamics. In this case the change point is said to be *exogenous* to the model. On the other side, events linked to the assets themselves can also cause a change in their dynamics, e.g. an asset price crossing a certain pre-specified level. Such a change point is said to be *endogenous* to the model. The nature of an event determines the dependence structure between the change point and the processes driving the asset price.

In this paper we study a general class of models with a change point, which are able to capture both of the above-mentioned situations. The dynamics of the asset price is modeled as a stochastic exponential of a continuous semimartingale, whose drift, volatility and driving Brownian motions change at the random time. Only minimal assumptions on the non-negative random variable mathematically representing the change point are imposed, in the sense that if such assumptions are violated, then the semimartingale property of the driving Brownian motions may be lost when changing filtrations and pathological forms of arbitrage are possible. The drift and the volatility are stochastic and depend on time and on the current state of the process and the two Brownian motions are not necessarily mutually independent. In particular, the random time is allowed to depend on the Brownian motions.

We aim at understanding the structure and the behavior of this class of models in all possible filtrations which naturally arise in this setting. These filtrations represent different levels of information, starting from the minimal knowledge of only the asset price process up to

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some time t to the full knowledge of the driving Brownian motions up to time t together with the knowledge of the random time already at time $t = 0$. Mathematically speaking, the various filtrations are obtained either as progressive or initial enlargements of a reference filtration with respect to the random time. Using enlargement of filtration techniques, we study the no-arbitrage properties of the model in each filtration. In particular, we characterize the absence of arbitrages of the first kind, which is equivalent to a square-integrability property of the market price of risk process. In turn, the latter condition allows us to obtain martingale representation theorems in the different filtrations. Combining these results we give for all filtrations a complete description of all equivalent local martingale measures and of market completeness. Even though we are not specifically concerned with the detection of the change point, we find that in the case when the two volatility regimes are distinct, the random time is actually a stopping time with respect to the price process filtration, hence there is no need to "detect" it. On the other side, if the two volatility regimes coincide everywhere, then the change point is not observable.

Our study of this type of models was inspired by a recent paper by Cawston and Vostrikova [5], where an exponential asset price model driven by two independent Lévy processes and with an independent change point is developed and analyzed in the context of utility maximization. In comparison with that paper, we refrain from imposing any independence assumption, giving a complete description of the model in all possible filtrations in a Brownian setting. We refer to Cawston and Vostrikova [5] for a comprehensive literature review on change point models, as well as their applications both in mathematical finance and in other areas. The focus in these papers is often on the problem of (quickest) detection of the change point. Change point models have been extensively studied in a series of papers by A. N. Shiryaev and co-authors, beginning from Shiryaev [39] to Shiryaev [40], where financial applications are discussed.

The model presented in this paper can also be seen as a regime switching model. By regime switching models we mean models in which the drift and the volatility of the price process are functions of a process taking finitely many values, which are interpreted for example as states of the economy. This underlying state process is usually assumed to be a Markov chain in order to ensure the analytical tractability of the model. A special case of our model, in which the two Brownian motions are assumed to be the same and the random time is the first jump time to an absorbing state 1 of a Markov process with two states $\{0, 1\}$, is interpreted as a regime switching model according to the above definition. Regime switching models have been widely employed for financial modeling, see for instance Elliott et al. [8], Elliott and Siu [11] and Goutte [15] for applications to the pricing and hedging of contingent claims and Elliott and Siu [10] in the context of portfolio optimization. We do not attempt here a detailed overview of the broad literature, but we mention the book Elliott and Mamon [9], which contains several applications to financial models under incomplete information, and the survey paper of Guo [16]. Regime switching models have been also widely studied from the statistical perspective, see for instance the seminal papers by Hamilton [17] and Quandt [37] as well as Chapter 22 of Hamilton [18] and the references therein.

Finally, a possible future research line is stochastic control of the asset price process in its own filtration, either in the manner of utility maximization like in, for example, Björk et al. [2], Lakner [33], Lakner [34] and Pham and Quenez [35], or extending a method lately introduced by Karatzas and Li [30].

The paper is organized as follows. In Section 2 we introduce the notation and the general setting of the model. Section 3 studies the properties of the model in the progressively enlarged filtrations $\mathbb{G}$ and $\mathbb{G}^X$. We also prove the well-posedness of the main SDE defining the model. In Section 4 we analyze the model in its own filtration $\mathbb{F}^X$. In particular, we study two special cases where the volatility functions differ or coincide everywhere, respectively. The distinct volatility functions allow for observing the random time τ, as in this case it is a stopping time with respect to the filtration $\mathbb{F}^X$. Section 5 is dedicated to the study of the properties of the model in the initially enlarged filtrations $\mathbb{G}^{(\tau)}$ and $\mathbb{G}^{X,(\tau)}$. Finally, Section 6 concludes by pointing out possible generalizations and applications.

2. General setting and preliminaries

We begin by introducing the main ingredients of the model together with some notation and definitions. Let $(\Omega, \mathcal{A}, P)$ be a given probability space, with P denoting the physical/statistical probability measure, and let $T \in (0, \infty)$ be a fixed time horizon. We assume that all random variables and stochastic processes introduced in the following are measurable with respect to the σ-fields $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{B}([0,T])$, respectively. Let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration on $(\Omega, \mathcal{A}, P)$, assumed to satisfy the usual conditions of right-continuity and P-completeness. For a given stochastic process $Y = (Y_t)_{0 \leq t \leq T}$ on $(\Omega, \mathcal{A}, P)$ we denote by $\mathbb{F}^Y = (\mathcal{F}_t^Y)_{0 \leq t \leq T}$ the right-continuous P-augmented natural filtration of Y.

The random change point is represented by a *random time* τ, i.e., an $\mathcal{A}$-measurable random variable $\tau : \Omega \to \mathbb{R}_+$ which is not necessarily an $\mathbb{F}$-stopping time. Furthermore, we let $W = (W_t)_{0 \leq t \leq T}$ and $W' = (W'_t)_{0 \leq t \leq T}$ be two Brownian motions on $(\Omega, \mathcal{A}, \mathbb{F}, P)$, with $[W, W']_t = \rho\, t$ for some correlation parameter $\rho \in [-1, 1]$.

We consider a financial market with one risky asset and one riskless asset. As usual in the literature, we take the riskless asset as the numï£¡raire and directly pass to discounted quantities. We denote by $S = (S_t)_{0 \leq t \leq T}$ the discounted price process of the risky asset and suppose that S can be represented as follows, for some fixed initial value $S_0 \in (0, \infty)$:

$$S = S_0\, \mathcal{E}(X), \tag{2.1}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential (with $\mathcal{E}(\cdot)_0 = 1$) and X is described by the SDE

$$\begin{aligned} \mathrm{d}X_t &= \big(\mathbf{1}_{\{t \leq \tau\}} \mu^1(t, X_t) + \mathbf{1}_{\{t > \tau\}} \mu^2(t, X_t)\big)\, \mathrm{d}t \\ &\qquad + \mathbf{1}_{\{t \leq \tau\}} \sigma^1(t, X_t)\, \mathrm{d}W_t + \mathbf{1}_{\{t > \tau\}} \sigma^2(t, X_t)\, \mathrm{d}W'_t, \\ X_0 &= 0, \end{aligned} \tag{2.2}$$

where $\mu^i : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma^i : [0,T] \times \mathbb{R} \to \mathbb{R}_+$, for $i = 1, 2$, are Borel-measurable functions with $\mu^i(t,x) \neq 0$ and $\sigma^i(t,x) \neq 0$, for all $(t,x) \in [0,T] \times \mathbb{R}$ and $i = 1, 2$.

Condition I. The functions $\mu^i : [0,T] \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ and $\sigma^i : [0,T] \times \mathbb{R} \to (0, \infty)$, for $i = 1, 2$, satisfy the following conditions:

(a) there exists a constant $K > 0$ such that:

$$|\mu^i(t,x) - \mu^i(t,y)| \leq K|x - y|, \qquad \forall t \in [0,T],\ \forall x, y \in \mathbb{R}, \ \text{ for } i = 1, 2;$$

$$|\sigma^i(t,x) - \sigma^i(t,y)| \leq K|x - y|, \qquad \forall t \in [0,T],\ \forall x, y \in \mathbb{R}, \ \text{ for } i = 1, 2;$$

(b) the function $(t,x) \mapsto \sigma^i(t,x)$ is jointly continuous in $(t,x) \in [0,T] \times \mathbb{R}$, for $i = 1, 2$.

Part (a) of Condition I consists of the usual global Lipschitz conditions on the functions μ^i and σ^i appearing in the SDE (2.2), while part (b) is needed for technical reasons.

Remarks 2.1. 1) As can be easily verified, part (a) of Condition I implies that there exists a constant $\bar{K} > 0$ such that the usual growth conditions hold:

$$|\mu^i(t,x)|^2 \leq \bar{K}\left(1+x^2\right), \qquad \forall t \in [0,T], \forall x \in \mathbb{R}, \text{ for } i=1,2;$$
$$|\sigma^i(t,x)|^2 \leq \bar{K}\left(1+x^2\right), \qquad \forall t \in [0,T], \forall x \in \mathbb{R}, \text{ for } i=1,2.$$

2) The process S has been specified in (2.1) as a stochastic exponential for notational convenience only. Indeed, we could equally work with ordinary exponentials, as in Cawston and Vostrikova [5]. This follows from the simple identity $\exp(X - X_0) = \mathcal{E}(X + [X]/2)$, whenever X is a continuous semimartingale.

3) Observe that $\mathbb{F}^S$ coincides with the filtration $\mathbb{F}^X$ generated by the process X, meaning that $\mathcal{F}_t^S = \mathcal{F}_t^X$ for all $t \in [0,T]$. Indeed, due to (2.1), it is evident that $\mathcal{F}_t^S \subseteq \mathcal{F}_t^X$. On the other hand, we have $X_t = x + \int_0^t S_u^{-1} \mathrm{d}S_u$ and, hence, we also have $\mathcal{F}_t^X \subseteq \mathcal{F}_t^S$ for all $t \in [0,T]$.

As mentioned in the introduction, we aim at studying the properties of the model (2.1)-(2.2) with respect to different levels of information, mathematically represented by different filtrations on $(\Omega, \mathcal{A}, P)$. In view of Remarks 2.1-3, we can and do restrict our attention to the study of the behavior of the process X in the following filtrations:

- **(i)** the filtration $\mathbb{F}^X$;
- **(ii)** the filtration $\mathbb{G}^X$, obtained as the *progressive enlargement* of $\mathbb{F}^X$ with respect to τ and defined as $\mathcal{G}_t^X := \bigcap_{s>t}\big(\mathcal{F}_s^X \vee \sigma(\tau \wedge s)\big)$ for all $t \in [0,T]$;
- **(iii)** the filtration $\mathbb{G}$, obtained as the *progressive enlargement* of $\mathbb{F}$ with respect to τ and defined as $\mathcal{G}_t := \bigcap_{s>t}\big(\mathcal{F}_s \vee \sigma(\tau \wedge s)\big)$ for all $t \in [0,T]$;
- **(iv)** the filtration $\mathbb{G}^{X,(\tau)}$, obtained as the *initial enlargement* of $\mathbb{F}^X$ with respect to τ and defined as $\mathcal{G}_t^{X,(\tau)} := \bigcap_{s>t}\big(\mathcal{F}_s^X \vee \sigma(\tau)\big)$ for all $t \in [0,T]$;
- **(v)** the filtration $\mathbb{G}^{(\tau)}$, obtained as the *initial enlargement* of $\mathbb{F}$ with respect to τ and defined as $\mathcal{G}_t^{(\tau)} := \bigcap_{s>t}\big(\mathcal{F}_s \vee \sigma(\tau)\big)$ for all $t \in [0,T]$.

The filtrations $\mathbb{G}^X$ and $\mathbb{G}$ are the smallest right-continuous filtrations which contain $\mathbb{F}^X$ and $\mathbb{F}$, respectively, and make τ a $\mathbb{G}^X$-stopping time and a $\mathbb{G}$-stopping time, respectively. For a detailed account of the theory of enlargement of filtrations we refer the reader to the monograph of Jeulin [28] and to Chapter VI of Protter [36]. It is easy to see that:

$$\mathbb{F}^X \subseteq \mathbb{G}^X \subseteq \mathbb{G} \subseteq \mathbb{G}^{(\tau)},$$

meaning that $\mathcal{F}_t^X \subseteq \mathcal{G}_t^X \subseteq \mathcal{G}_t \subseteq \mathcal{G}_t^{(\tau)}$ for all $t \in [0,T]$. It is also evident that

$$\mathbb{G}^X \subseteq \mathbb{G}^{X,(\tau)} \subseteq \mathbb{G}^{(\tau)}.$$

Intuitively, in the special case where $\mathbb{F} = \mathbb{F}^{W^1} \vee \mathbb{F}^{W^2}$, the different filtrations introduced above correspond to economic agents having access to different levels of information:

- **(i)** $\mathcal{F}_t^X$: the knowledge of only the price process of the risky asset up to time t;
- **(ii)** $\mathcal{G}_t^X$: the knowledge of the price process of the risky asset up to time t plus the knowledge of the random time τ if the latter has occurred before time t;
- **(iii)** $\mathcal{G}_t$: the knowledge of the two driving Brownian motions W^1 and W^2 up to time t plus the knowledge of the random time τ if the latter has occurred before time t;

(iv) $\mathcal{G}_t^{X,(\tau)}$: the knowledge of the price process of the risky asset up to time t plus the knowledge (already at time $t=0$) of the random time τ;

(v) $\mathcal{G}_t^{(\tau)}$: the knowledge of the two driving Brownian motions W^1 and W^2 up to time t plus the knowledge (already at time $t=0$) of the random time τ.

In the remaining part of the paper, we shall denote by ${}^pY$ the predictable projection of a given process $Y=(Y_t)_{0\le t\le T}$ onto one of the filtrations introduced above (see e.g. He et al. [19], Section V.1). The filtration onto which we will take projections changes throughout the paper, but this will be made clear in the text.

Let us denote by $\mathbb{A}=(\mathcal{A}_t)_{0\le t\le T}$ a generic filtration on $(\Omega,\mathcal{A},P)$ with respect to which the process S is a semimartingale and let $L(S,\mathbb{A})$ be the set of all S-integrable $\mathbb{A}$-predictable processes, in the sense of Definition 9.13 in He et al. [19]. We denote by $\int h\,\mathrm{d}S$ the stochastic integral process $\left(\int_0^t h_u\,\mathrm{d}S_u\right)_{0\le t\le T}$, for $h=(h_t)_{0\le t\le T}\in L(S,\mathbb{A})$, and by $\mathcal{M}_{\mathrm{loc}}(\mathbb{A})$ the family of all $\mathbb{A}$-local martingales. Note that, if $S=M+B$ denotes the canonical decomposition of S into a continuous $\mathbb{A}$-local martingale M and a continuous $\mathbb{A}$-predictable process of finite variation B, we have $\int h\,\mathrm{d}S=\int h\,\mathrm{d}M+\int h\,\mathrm{d}B$ (see He et al. [19], Theorem 9.16). We recall two important notions of arbitrage which have been considered in the literature.

Definition 2.2.

(i) A non-negative $\mathcal{A}_T$-measurable random variable ζ is said to yield an *Arbitrage of the First Kind in* $\mathbb{A}$ if $P(\zeta>0)>0$ and if for every $v\in(0,\infty)$ there exists an element $h^v\in L(S,\mathbb{A})$ such that $v+\int h^v\mathrm{d}S\ge 0$ P-a.s. and $v+\int_0^T h_t^v\mathrm{d}S_t\ge\zeta$ P-a.s. If there exists no such random variable we say that the *No Arbitrage of the First Kind (NA1)* condition holds for $\mathbb{A}$.

(ii) A sequence $\{h^n\}_{n\in\mathbb{N}}\subset L(S,\mathbb{A})$ such that $\int h^n\mathrm{d}S$ is P-a.s. bounded from below, for all $n\in\mathbb{N}$, is said to yield a *Free Lunch with Vanishing Risk in* $\mathbb{A}$ if there exist an $\varepsilon>0$ and an increasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with $0\le\delta_n\nearrow 1$ such that $\int_0^T h_t^n\mathrm{d}S_t>-1+\delta_n$ P-a.s. and $P\left(\int_0^T h_t^n\mathrm{d}S_t>\varepsilon\right)\ge\varepsilon$, for all $n\in\mathbb{N}$. If there exists no such sequence we say that the *No Free Lunch with Vanishing Risk (NFLVR)* condition holds for $\mathbb{A}$.

The notion of *Arbitrage of the First Kind* has been introduced by Kardaras [31] and is stronger than the classical notion of *Free Lunch with Vanishing Risk*, which goes back to Delbaen and Schachermayer [6]. In particular, Proposition 1 of Kardaras [31] and Proposition 4.19 of Karatzas and Kardaras [29] imply that NA1 can be regarded as the minimal condition for market viability. We refer to Fontana and Runggaldier [14] for a study of the two no-arbitrage conditions introduced above in the context of general diffusion-based models.

We close this section by rewriting the SDE (2.2) in a more convenient way. We define the processes $W^1=(W_t^1)_{0\le t\le T}$ and $W^2=(W_t^2)_{0\le t\le T}$ by:

$$
\begin{aligned}
W_t^1 &:= W_t,\\
W_t^2 &:= \begin{cases}(W_t'-\rho W_t)/\sqrt{1-\rho^2} & \text{if } |\rho|\neq 1;\\ \rho W_t & \text{if } |\rho|=1.\end{cases}
\end{aligned}
$$

If $|\rho| \neq 1$, it is well known that W^1 and W^2 are two independent Brownian motions on $(\Omega, \mathcal{A}, \mathbb{F}, P)$. The SDE (2.2) can be equivalently rewritten as:

$$\begin{aligned} \mathrm{d}X_t &= \big(\mathbf{1}_{\{t\leq\tau\}}\mu^1(t,X_t) + \mathbf{1}_{\{t>\tau\}}\mu^2(t,X_t)\big)\,\mathrm{d}t \\ &\quad + \mathbf{1}_{\{t\leq\tau\}}\sigma^1(t,X_t)\,\mathrm{d}W^1_t + \mathbf{1}_{\{t>\tau\}}\sigma^2(t,X_t)\left(\rho\,\mathrm{d}W^1_t + \sqrt{1-\rho^2}\,\mathrm{d}W^2_t\right) \\ X_0 &= 0. \end{aligned} \tag{2.3}$$

The existence and uniqueness of a solution to the above SDE on a suitable filtered probability space will be proved in Section 3.1 (see Proposition 3.1).

3. The progressively enlarged filtrations $\mathbb{G}$ and $\mathbb{G}^X$

In this section we study the progressively enlarged filtrations $\mathbb{G}$ and $\mathbb{G}^X$. We shall make no assumption on the random time τ apart from a very weak semimartingale-preservation hypothesis (Condition II). We start our analysis with the progressively enlarged filtration $\mathbb{G}$, which is easier to describe than the filtration $\mathbb{G}^X$. Moreover, starting with the filtration $\mathbb{G}$ allows us to prove the well-posedness of the SDE (2.3).

3.1. **The progressively enlarged filtration $\mathbb{G}$.** The filtration $\mathbb{G}$ is the smallest filtration satisfying the usual conditions which contains $\mathbb{F}$ and makes τ a $\mathbb{G}$-stopping time. However, the semimartingale property is not necessarily preserved when passing from the filtration $\mathbb{F}$ to the filtration $\mathbb{G}$. In particular, the Brownian motions W^1 and W^2 may fail to be $\mathbb{G}$-semimartingales. The following condition prevents this pathological behavior[1].

Condition II. There exist two $\mathbb{G}$-predictable processes $\theta^1 = (\theta^1_t)_{0\leq t\leq T}$ and $\theta^2 = (\theta^2_t)_{0\leq t\leq T}$ and two $\mathbb{G}$-Brownian motions $\widetilde{W}^1 = (\widetilde{W}^1_t)_{0\leq t\leq T}$ and $\widetilde{W}^2 = (\widetilde{W}^2_t)_{0\leq t\leq T}$ such that:

$$W^i_t = \widetilde{W}^i_t + \int_0^t \theta^i_u\,\mathrm{d}u, \qquad \text{for all } t\in[0,T] \text{ and for } i=1,2.$$

Condition II can be regarded as a rather weak form of the *($\mathcal{H}'$)-hypothesis* from the theory of enlargement of filtrations, which assumes that all $\mathbb{F}$-semimartingales are also $\mathbb{G}$-semimartingales (Jeulin [28], Chapter II). Condition II can be shown to hold for almost all random time models considered in financial and insurance mathematics. For instance, Condition II is trivially satisfied (with $\theta^i \equiv 0$, for $i=1,2$) if the random time τ is obtained through the *canonical construction*, see e.g. Section 8.2.1 of Bielecki and Rutkowski [1]. Condition II is also satisfied in many other cases, including τ being an *honest time* (see Jeulin [28], Chapter V), random times which satisfy the *density hypothesis* (see El Karoui et al. [7] and Jeanblanc and Le Cam [23]) or random times constructed through the approach introduced in Jeanblanc and Song [25, 26].

3.1.1. *Existence and uniqueness of the solution to the SDE* (2.3). As long as Condition II holds, equation (2.3) makes sense as a semimartingale-driven SDE on $(\Omega, \mathcal{A}, \mathbb{G}, P)$. This provides a good setting for establishing the existence and uniqueness of a solution. We say that a $\mathbb{G}$-semimartingale $X = (X_t)_{0\leq t\leq T}$ is a *solution* to the SDE (2.3) on $(\Omega, \mathcal{A}, \mathbb{G}, P)$ if $X_0 = 0$ and X satisfies equation (2.3) with respect to the $\mathbb{G}$-semimartingales W^1 and

[1]We want to point out that, due to Proposition 4.16 of Jeulin [28] together with the Kunita-Watanabe inequality, Condition II is always satisfied up to the $\mathbb{G}$-stopping time τ.

W^2. This corresponds to the notion of *strong solution* of a semimartingale-driven SDE, as considered in Chapter V of Protter [36] (see also Jacod [21], Chapter XIV).

Proposition 3.1. *Suppose that Conditions I and II hold. Then there exists a unique continuous $\mathbb{G}$-semimartingale $X = (X_t)_{0\leq t\leq T}$ which is a solution to the SDE* (2.3) *on* $(\Omega, \mathcal{A}, \mathbb{G}, P)$.

Proof. Since τ is a $\mathbb{G}$-stopping time, the processes $\mathbf{1}_{[\![0,\tau]\!]}$ and $\mathbf{1}_{(\!(\tau,T]\!]}$ are $\mathbb{G}$-predictable, being $\mathbb{G}$-adapted and left-continuous, and admit limits from the right. Let us define the following random functions, for $\omega \in \Omega$, $t \in [0, T]$, $x \in \mathbb{R}$:

$$\begin{aligned} g(\omega, t, x) &:= \mathbf{1}_{\{t\leq\tau(\omega)\}}\mu^1(t, x) + \mathbf{1}_{\{t>\tau(\omega)\}}\mu^2(t, x);\\ f(\omega, t, x) &:= \mathbf{1}_{\{t\leq\tau(\omega)\}}\sigma^1(t, x) + \mathbf{1}_{\{t>\tau(\omega)\}}\sigma^2(t, x). \end{aligned} \tag{3.1}$$

Condition I implies that f and g are random Lipschitz, in the sense of Protter [36], page 256. The existence of a unique solution $X = (X_t)_{0\leq t\leq T}$ to the SDE (2.3) on $(\Omega, \mathcal{A}, \mathbb{G}, P)$ then follows from Theorem V.6 of Protter [36]. □

Remark 3.2. Note that, under the assumptions of Proposition 3.1, the joint continuity of the function $\sigma^i : [0, T] \times \mathbb{R} \to (0, +\infty)$, for $i = 1, 2$, together with the continuity of X implies that the random variable $\xi := \min_{t\in[0,T]}\{\sigma^1(t, X_t) \wedge \sigma^2(t, X_t)\}$ is well-defined, P-a.s. finite and strictly positive.

3.1.2. *Canonical decomposition and no-arbitrage properties in* $\mathbb{G}$. Let us now investigate the no-arbitrage properties of the financial market where the asset S is traded with respect to the information contained in the progressively enlarged filtration $\mathbb{G}$. As a preliminary, we write the canonical decomposition of the process $X = (X_t)_{0\leq t\leq T}$ in the filtration $\mathbb{G}$:

$$X_t = \int_0^t \tilde{\mu}_u \,\mathrm{d}u + \int_0^t V_u \,\mathrm{d}\widetilde{W}_u, \qquad \text{for all } t \in [0, T], \tag{3.2}$$

where the processes $\tilde{\mu} = (\tilde{\mu}_t)_{0\leq t\leq T}$, $V = (V_t)_{0\leq t\leq T}$ and $\widetilde{W} = (\widetilde{W}_t)_{0\leq t\leq T}$ are defined as[2]:

$$\tilde{\mu}_t := \mathbf{1}_{\{t\leq\tau\}} \left(\mu^1(t, X_t) + \sigma^1(t, X_t)\theta^1_t\right) + \mathbf{1}_{\{t>\tau\}} \left(\mu^2(t, X_t) + \sigma^2(t, X_t)\big(\rho\theta^1_t + \sqrt{1-\rho^2}\theta^2_t\big)\right); \tag{3.3}$$

$$V_t := \mathbf{1}_{\{t\leq\tau\}}\sigma^1(t, X_t) + \mathbf{1}_{\{t>\tau\}}\sigma^2(t, X_t); \tag{3.4}$$

$$\begin{aligned} \widetilde{W}_t &:= \mathbf{1}_{\{t\leq\tau\}}\widetilde{W}^1_t + \mathbf{1}_{\{t>\tau\}} \left(\rho\widetilde{W}^1_t + (1-\rho)\widetilde{W}^1_\tau + \sqrt{1-\rho^2}\left(\widetilde{W}^2_t - \widetilde{W}^2_\tau\right)\right)\\ &= \int_0^{t\wedge\tau} \mathrm{d}\widetilde{W}^1_u + \rho\int_\tau^{t\vee\tau} \mathrm{d}\widetilde{W}^1_u + \sqrt{1-\rho^2}\int_\tau^{t\vee\tau} \mathrm{d}\widetilde{W}^2_u. \end{aligned} \tag{3.5}$$

Since $[\widetilde{W}]_t = t$, for all $t \in [0, T]$, the continuous $\mathbb{G}$-local martingale $\widetilde{W}$ is a $\mathbb{G}$-Brownian motion. Equation (3.2) gives the canonical decomposition of X with respect to $\mathbb{G}$ and allows us to formulate the next proposition, which characterizes the no-arbitrage properties of the model (2.1)-(2.3) when considered in the progressively enlarged filtration $\mathbb{G}$. We define the $\mathbb{G}$-predictable process $\bar{\theta} = (\bar{\theta}_t)_{0\leq t\leq T}$ as:

$$\bar{\theta} := \mathbf{1}_{[\![0,\tau]\!]}\theta^1 + \mathbf{1}_{(\!(\tau,T]\!]}\big(\rho\theta^1 + \sqrt{1-\rho^2}\theta^2\big). \tag{3.6}$$

[2]Note that Condition II implicitly requires that $\int_0^t|\theta^i_u|\mathrm{d}u < \infty$ P-a.s., for all $t \in [0, T]$ and $i = 1, 2$. In turn, due to Condition I-(b), this implies that $\int_0^t\theta^i_u\,\sigma^i(u, X_u)\,\mathrm{d}u$ is well-defined, for all $t \in [0, T]$ and $i = 1, 2$.

Proposition 3.3. *Suppose that Conditions I and II hold. Then the following assertions hold for the model* (2.1)-(2.3) *considered with respect to the filtration* $\mathbb{G}$*:*

(a) *NA1 holds if and only if* $\int_0^T \bar{\theta}_t^2 \,\mathrm{d}t < \infty$ P*-a.s., with the latter condition being equivalent to* $\int_0^T (\tilde{\mu}_t/V_t)^2 \,\mathrm{d}t < \infty$ P*-a.s.;*

(b) *NFLVR holds if and only if NA1 holds and there exists* $N = (N_t)_{0\leq t\leq T} \in \mathcal{M}_{loc}(\mathbb{G})$ *with* $N_0 = 0$, $\Delta N > -1$ P*-a.s.,* $[N, \widetilde{W}] = 0$ *such that* $E\big[\mathcal{E}(-\int(\tilde{\mu}/V)\,\mathrm{d}\widetilde{W} + N)_T\big] = 1$.

Proof. Note first that, due to Conditions I-II, the process $\tilde{\mu}/V$ is well-defined. Theorem 4 of Kardaras [31] implies that NA1 holds if and only if the following condition holds:

$$\int_0^T \left(\frac{\tilde{\mu}_t}{V_t}\right)^2 \mathrm{d}t = \int_0^{T\wedge\tau} \left(\frac{\mu^1(t,X_t)}{\sigma^1(t,X_t)} + \theta_t^1\right)^2 \mathrm{d}t + \int_\tau^{T\vee\tau} \left(\frac{\mu^2(t,X_t)}{\sigma^2(t,X_t)} + \rho\,\theta_t^1 + \sqrt{1-\rho^2}\,\theta_t^2\right)^2 \mathrm{d}t < \infty \ P\text{-a.s.}$$

By using the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ together with Remarks 2.1-1 and 3.2 we can write:

$$\begin{aligned}
\int_0^T \left(\frac{\tilde{\mu}_t}{V_t}\right)^2 \mathrm{d}t &\leq 2\left(\int_0^{T\wedge\tau} \left(\frac{\mu^1(t,X_t)}{\sigma^1(t,X_t)}\right)^2 \mathrm{d}t + \int_\tau^{T\vee\tau} \left(\frac{\mu^2(t,X_t)}{\sigma^2(t,X_t)}\right)^2 \mathrm{d}t\right) + 2\int_0^T \bar{\theta}_t^2 \,\mathrm{d}t \\
&\leq \frac{2\bar{K}}{\xi^2}\int_0^T (1+X_t^2)\,\mathrm{d}t + 2\int_0^T \bar{\theta}_t^2\,\mathrm{d}t \leq \frac{2\bar{K}T}{\xi^2}\Big(1 + \max_{t\in[0,T]} X_t^2\Big) + 2\int_0^T \bar{\theta}_t^2\,\mathrm{d}t.
\end{aligned}$$

Analogously, using the elementary inequality $(a+b)^2 \geq b^2/2 - a^2$:

$$\begin{aligned}
\int_0^T \left(\frac{\tilde{\mu}_t}{V_t}\right)^2 \mathrm{d}t &\geq \frac{1}{2}\int_0^T \bar{\theta}_t^2\,\mathrm{d}t - \int_0^{T\wedge\tau} \left(\frac{\mu^1(t,X_t)}{\sigma^1(t,X_t)}\right)^2 \mathrm{d}t - \int_\tau^{T\vee\tau} \left(\frac{\mu^2(t,X_t)}{\sigma^2(t,X_t)}\right)^2 \mathrm{d}t \\
&\geq \frac{1}{2}\int_0^T \bar{\theta}_t^2\,\mathrm{d}t - \frac{\bar{K}T}{\xi^2}\Big(1 + \max_{t\in[0,T]} X_t^2\Big).
\end{aligned}$$

Since X is continuous, the above inequalities show that

$$\int_0^T (\tilde{\mu}_t/V_t)^2\,\mathrm{d}t < \infty \ P\text{-a.s.} \quad \text{if and only if} \quad \int_0^T \bar{\theta}_t^2\,\mathrm{d}t < \infty \ P\text{-a.s.}$$

thus proving part (a). To prove part (b), recall that NFLVR holds if and only if there exists a strictly positive $\mathbb{G}$-martingale $Z = (Z_t)_{0\leq t\leq T}$ such that $ZS \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G})$ (see Delbaen and Schachermayer [6], Corollary 1.2). Due to (2.1), the latter holds if and only if $ZX \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G})$. Hence, using the terminology of Fontana and Runggaldier [14], NFLVR holds if and only if there exists a martingale deflator Z for X with $E[Z_T] = 1$. Part (b) then follows from (3.2) together with Lemma 4.3.15 of Fontana and Runggaldier [14] (compare also with Schweizer [38], Theorem 1). □

Remark 3.4. Note that, if the Brownian motions W^1 and W^2 are $\mathbb{G}$-semimartingales but their finite variation parts are not absolutely continuous with respect to dt (i.e., Condition II is violated), one can then obtain the most egregious form of arbitrage (i.e., an *increasing profit*) in the filtration $\mathbb{G}$, see e.g. Section 4.3 of Fontana and Runggaldier [14]. In this sense, Condition II is minimal for the study of the no-arbitrage properties of the model (2.1)-(2.3).

Proposition 3.3 shows that the process $\bar{\theta}$ defined in (3.6) plays a crucial role in determining the no-arbitrage properties of the model (2.1)-(2.3) when considered in the filtration $\mathbb{G}$. In turn, this implies that the existence of arbitrages in $\mathbb{G}$ crucially depends on the properties of τ. For instance, the condition $\int_0^T \bar{\theta}_t^2 \,\mathrm{d}t < \infty$ P-a.s. may fail if τ is an *honest time* with $P(\tau < T) = 1$, as demonstrated in Imkeller [20] and Fontana et al. [13].

Remark 3.5 (*The $\mathbb{G}$-martingale representation property*)**.** Let us denote by $A^{\mathbb{G}}$ the $\mathbb{G}$-predictable compensator of the increasing process $(\mathbf{1}_{\{\tau\leq t\}})_{0\leq t\leq T}$ (see Protter [36], Section III.5) and let $M^{\mathbb{G}} := \mathbf{1}_{\{\tau\leq\cdot\}} - A^{\mathbb{G}}$ be the corresponding compensated $\mathbb{G}$-martingale. Suppose that every $\mathbb{G}$-local martingale $L = (L_t)_{0\leq t\leq T}$ admits the representation:

$$L_t = L_0 + \int_0^t \phi_u^1 \,\mathrm{d}\widetilde{W}_u^1 + \int_0^t \phi_u^2 \,\mathrm{d}\widetilde{W}_u^2 + \int_0^t \psi_u \,\mathrm{d}M_u^{\mathbb{G}}, \qquad \text{for all } t \in [0,T], \tag{3.7}$$

where $\phi^i = (\phi_t^i)_{0\leq t\leq T}$ is a $\mathbb{G}$-predictable process with $\int_0^T (\phi_t^i)^2 \mathrm{d}t < \infty$ P-a.s., for $i = 1,2$, and $\psi = (\psi_t)_{0\leq t\leq T}$ is a $\mathbb{G}$-predictable process with $\int_0^T |\psi_t||\mathrm{d}A_t^{\mathbb{G}}| < \infty$ P-a.s. and where $\widetilde{W}^i$, for $i = 1,2$, is the $\mathbb{G}$-Brownian motion introduced in Condition II. In this case, we can obtain a more precise description of the $\mathbb{G}$-local martingale $N = (N_t)_{0\leq t\leq T}$ appearing in part (b) of Proposition 3.3. Indeed, using (3.2)-(3.5) together with (3.7), we get:

$$N_t = \int_0^{t\wedge\tau} \varphi_u^2 \,\mathrm{d}\widetilde{W}_u^2 + \mathbf{1}_{\{\rho=0\}} \int_\tau^{t\vee\tau} \varphi_u^1 \,\mathrm{d}\widetilde{W}_u^1 + \mathbf{1}_{\{\rho\neq0\}} \left(\int_\tau^{t\vee\tau} \varphi_u^3 \,\mathrm{d}\widetilde{W}_u^2 - \frac{\sqrt{1-\rho^2}}{\rho} \int_\tau^{t\vee\tau} \varphi_u^3 \,\mathrm{d}\widetilde{W}_u^1 \right) + \int_0^t \psi_u \,\mathrm{d}M_u^{\mathbb{G}}, \tag{3.8}$$

where φ^1, φ^2, φ^3 and ψ are arbitrary $\mathbb{G}$-predictable integrable processes. Summing up, as soon as NFLVR and the martingale representation property (3.7) hold in the filtration $\mathbb{G}$, part (b) of Proposition 3.3 and (3.8) give a complete characterization of all equivalent local martingale measures in $\mathbb{G}$. If $\mathbb{F} = \mathbb{F}^{W^1} \vee \mathbb{F}^{W^2}$, necessary and sufficient conditions for the validity of the martingale representation property (3.7) have been recently established by Jeanblanc and Song [27]. In particular, the representation (3.7) always holds if τ is obtained through the *canonical construction*, if τ satisfies the *density hypothesis* or if τ is constructed as in Jeanblanc and Song [26].

3.2. **The progressively enlarged price process filtration $\mathbb{G}^X$.** This section studies the properties of the model (2.1)-(2.3) when considered with respect to the filtration $\mathbb{G}^X$.

3.2.1. *Canonical decomposition and no-arbitrage properties in $\mathbb{G}^X$.* The next lemma gives the canonical decomposition of X with respect to the filtration $\mathbb{G}^X$. The idea of the proof is inspired by the reduction of partial observation control problems into those with full observations and consists of projecting the canonical decomposition (3.2) obtained with respect to $\mathbb{G}$ onto the smaller filtration $\mathbb{G}^X$. In order to take care of integrability issues, a localization procedure is needed.

Lemma 3.6. *Suppose that Conditions I and II hold. Then the process X admits the following canonical decomposition with respect to the filtration $\mathbb{G}^X$:*

$$X_t = \int_0^t \mu_u \,\mathrm{d}u + \int_0^t V_u \,\mathrm{d}B_u, \qquad \textit{for all } t \in [0,T], \tag{3.9}$$

where the $\mathbb{G}^X$-predictable process $\mu = (\mu_t)_{0\leq t\leq T}$ is defined as:

$$\mu_t := \mathbf{1}_{\{t\leq\tau\}}\bigl(\mu^1(t,X_t)+\sigma^1(t,X_t)\,{}^p\theta^1_t\bigr)+\mathbf{1}_{\{t>\tau\}}\Bigl(\mu^2(t,X_t)+\sigma^2(t,X_t)\bigl(\rho\,{}^p\theta^1_t+\sqrt{1-\rho^2}\,{}^p\theta^2_t\bigr)\Bigr), \tag{3.10}$$

with ${}^p\theta^i$ denoting the $\mathbb{G}^X$-predictable projection of θ^i, for $i=1,2$, and where the process $B=(B_t)_{0\leq t\leq T}$ is a $\mathbb{G}^X$-Brownian motion and the $\mathbb{G}^X$-predictable process $V=(V_t)_{0\leq t\leq T}$ is defined as in (3.4).

Proof. The unique solution $X=(X_t)_{0\leq t\leq T}$ to the SDE (2.3) on $(\Omega,\mathcal{A},\mathbb{G},P)$ is a continuous $\mathbb{G}$-semimartingale. Hence, the result of Stricker [41] implies that it is also a $\mathbb{G}^X$-semimartingale, since $\mathbb{G}^X\subset\mathbb{G}$. Let $X=A+M$ be the canonical decomposition of X in $\mathbb{G}^X$, where A is a continuous $\mathbb{G}^X$-predictable process of finite variation with $A_0=0$ and $M\in\mathcal{M}_{\rm loc}(\mathbb{G}^X)$ with $M_0=0$. For every $n\in\mathbb{N}$, define the $\mathbb{G}^X$-stopping time $\tau_n:=\inf\{t\in[0,T]:|X_t|\geq n\}\wedge T$. Clearly, we have $\tau_n\nearrow T$ P-a.s. as $n\to\infty$. Due to Remark 2.1-1, there exists a constant $\bar{K}>0$ such that, for every $n\in\mathbb{N}$:

$$\int_0^{\tau_n\wedge T} V_t^2\,\mathrm{d}t\leq\bar{K}\int_0^{\tau_n\wedge T}\bigl(1+X_t^2\bigr)\,\mathrm{d}t\leq\bar{K}(1+n^2)\,T.$$

Since $[M]_t=\int_0^t V_u^2\,\mathrm{d}u$, for all $t\in[0,T]$, this shows that $\{\tau_n\}_{n\in\mathbb{N}}$ is a $\mathbb{G}^X$-localizing sequence for the $\mathbb{G}^X$-local martingale M as well as for the $\mathbb{G}$-local martingale $\int V\mathrm{d}\widetilde{W}$. Proposition 9.24 of Jacod [21] together with (3.2) then implies that A is given by the dual $\mathbb{G}^X$-predictable projection of the process $\int_0^\cdot\tilde{\mu}_t\,\mathrm{d}t$, i.e. $A=\bigl(\int_0^\cdot\tilde{\mu}_t\,\mathrm{d}t\bigr)^p$. Furthermore, as is shown in 1.40 of Jacod [21], we have $\bigl(\int_0^\cdot\tilde{\mu}_t\,\mathrm{d}t\bigr)^p=\int_0^\cdot{}^p\tilde{\mu}_t\,\mathrm{d}t$, where ${}^p\tilde{\mu}$ denotes the $\mathbb{G}^X$-predictable projection of $\tilde{\mu}$. Equation (3.10) then follows by noting that, since τ is a $\mathbb{G}^X$-stopping time, the processes $\mathbf{1}_{[\![0,\tau]\!]}$ and $\mathbf{1}_{(\!(\tau,T]\!]}$ are $\mathbb{G}^X$-adapted and left-continuous and, hence, $\mathbb{G}^X$-predictable. To finish the proof, the process V defined in (3.4) never hits zero and is $\mathbb{G}^X$-predictable, τ being a $\mathbb{G}^X$-stopping time. This implies that the stochastic integral $B:=\int V^{-1}\,\mathrm{d}M$ is well-defined as a continuous $\mathbb{G}^X$-local martingale with $B_0=0$. Since $[B]_t=\int_0^t V_u^{-2}\,\mathrm{d}[M]_u=t$, for all $t\in[0,T]$, Lï£¡vy's characterization theorem allows to conclude that B is a $\mathbb{G}^X$-Brownian motion. □

Remark 3.7. We can equivalently define the process μ by using $\mathbb{G}^X$-*optional* projections instead of $\mathbb{G}^X$-*predictable* projections. This is due to the fact that optional and predictable projections differ only at countably many stopping times (see He et al. [19], Theorem 5.5).

We can now answer the question of whether the model (2.1)-(2.3), considered with respect to the filtration $\mathbb{G}^X$, satisfies the NA1 condition. We denote by ${}^p\bar{\theta}$ the $\mathbb{G}^X$-predictable projection of the process $\bar{\theta}=\mathbf{1}_{[\![0,\tau]\!]}\theta^1+\mathbf{1}_{(\!(\tau,T]\!]}\bigl(\rho\,\theta^1+\sqrt{1-\rho^2}\,\theta^2\bigr)$ introduced in (3.6).

Proposition 3.8. *Suppose that Conditions I and II hold. Then the following assertions hold for the model* (2.1)-(2.3) *considered with respect to the filtration $\mathbb{G}^X$:*

(a) *NA1 holds if and only if $\int_0^T{}^p\bar{\theta}_t^2\,\mathrm{d}t<\infty$ P-a.s., with the latter condition being equivalent to $\int_0^T(\mu_t/V_t)^2\,\mathrm{d}t<\infty$ P-a.s.*
(b) *NFLVR holds if and only if NA1 holds and there exists $N=(N_t)_{0\leq t\leq T}\in\mathcal{M}_{\rm loc}(\mathbb{G}^X)$ with $N_0=0$, $\Delta N>-1$ P-a.s., $[N,B]=0$ such that $E\bigl[\mathcal{E}\bigl(-\int(\mu/V)\mathrm{d}B+N\bigr)_T\bigr]=1$.*

Proof. By relying on the same arguments used in the proof of part (a) of Proposition 3.3, NA1 holds in $\mathbb{G}^X$ if and only if $\int_0^T (\mu_t/V_t)^2\,\mathrm{d}t < \infty$ P-a.s. and the latter condition is verified if and only if $\int_0^T {}^p\bar{\theta}_t^2\,\mathrm{d}t < \infty$ P-a.s. Then, as in the proof of part (b) of Proposition 3.3, NFLVR holds in $\mathbb{G}^X$ if and only if there exists a strictly positive $\mathbb{G}^X$-martingale Z such that $ZX \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}^X)$. Part (b) then follows from Lemma 3.6 together with Lemma 4.3.15 of Fontana and Runggaldier [14]. □

In particular, Proposition 3.8 shows that the validity of the NA1 condition for the model (2.1)-(2.3) in the filtration $\mathbb{G}^X$ crucially depends on the process ${}^p\bar{\theta}$, which in turn depends on the properties of the random time τ.

3.2.2. *Martingale representation property in $\mathbb{G}^X$.* We now study in more detail the structure of the filtration $\mathbb{G}^X$. In particular, we aim at proving an interesting martingale representation result (see Proposition 3.10). In turn, this will lead to an explicit characterization of all equivalent local martingale measures for the model (2.1)-(2.3) when considered with respect to the filtration $\mathbb{G}^X$ (see Corollary 3.11).

As a preliminary, observe that the process $1/V$ is well-defined, $\mathbb{G}^X$-predictable and locally bounded, being left-continuous by part (b) of Condition I. Hence, we can define the $\mathbb{G}^X$-adapted continuous process $\widehat{Y} = (\widehat{Y}_t)_{0\leq t\leq T}$ as follows, for all $t \in [0,T]$:

$$\widehat{Y}_t := \int_0^t V_u^{-1}\,\mathrm{d}X_u = \int_0^t \frac{\mu_u}{V_u}\,\mathrm{d}u + B_t, \tag{3.11}$$

where the processes μ and B are as in Lemma 3.6. Let us denote by $\mathbb{F}^{\widehat{Y}} = (\mathcal{F}_t^{\widehat{Y}})_{0\leq t\leq T}$ the right-continuous P-augmented natural filtration of $\widehat{Y}$ and by $\mathbb{G}^{\widehat{Y}} = (\mathcal{G}_t^{\widehat{Y}})_{0\leq t\leq T}$ the progressive enlargement of $\mathbb{F}^{\widehat{Y}}$ with respect to τ, meaning that $\mathcal{G}_t^{\widehat{Y}} = \bigcap_{s>t}\bigl(\mathcal{F}_s^{\widehat{Y}} \vee \sigma(\tau\wedge s)\bigr)$ for all $t \in [0,T]$. We can now prove a useful lemma which describes the structure of the filtration $\mathbb{G}^X$, showing that it coincides with the progressive enlargement with respect to τ of the filtration generated by the drifted Brownian motion $\widehat{Y}$.

Lemma 3.9. *Suppose that Conditions I and II hold. Then $\mathbb{G}^X = \mathbb{G}^{\widehat{Y}}$.*

Proof. Clearly, the process $\widehat{Y}$ defined in (3.11) is $\mathbb{G}^X$-adapted and τ is a $\mathbb{G}^X$-stopping time. This implies that $\mathbb{G}^{\widehat{Y}} \subseteq \mathbb{G}^X$. To prove the converse inclusion, let us first note that the process X can be represented as follows, where the random function f is defined as in (3.1):

$$X_t = \int_0^t f(u, X_u)\,\mathrm{d}\widehat{Y}_u, \qquad \text{for all } t \in [0,T].$$

Let us define the process $X^0 = (X_t^0)_{0\leq t\leq T}$ by $X_t^0 := 0$ for all $t \in [0,T]$ and define inductively, for every $k \in \mathbb{N}$, the process $X^k = (X_t^k)_{0\leq t\leq T}$ as:

$$X_t^{k+1} := \int_0^t f(u, X_u^k)\,\mathrm{d}\widehat{Y}_u, \qquad \text{for all } t \in [0,T].$$

By construction, for every $k \in \mathbb{N}$, the process X^k is adapted to the filtration $\mathbb{G}^{\widehat{Y}}$. Furthermore, since the function f is random Lipschitz, Theorem V.8 of Protter [36] implies that the process X^k converges to X uniformly on compacts in probability and, up to a subsequence, the

convergence takes place P-a.s. uniformly on compacts. This implies that X is adapted to the filtration $\mathbb{G}^{\widehat{Y}}$, thus showing that $\mathbb{G}^X \subseteq \mathbb{G}^{\widehat{Y}}$. $\square$

We are now in position to prove a martingale representation result for the filtration $\mathbb{G}^X$. It is important to note that we do not make any assumption on the random time τ nor on the underlying filtration $\mathbb{F}$ (in particular, we do not assume that $\mathbb{F} = \mathbb{F}^{W^1} \vee \mathbb{F}^{W^2}$). We denote by $A^{\mathbb{G}^X}$ the $\mathbb{G}^X$-predictable compensator of τ and by $M^{\mathbb{G}^X} := \mathbf{1}_{\{\tau \leq \cdot\}} - A^{\mathbb{G}^X}$ the associated compensated $\mathbb{G}^X$-martingale[3].

Proposition 3.10. *Suppose that Conditions I and II hold and assume in addition that NA1 holds in the filtration $\mathbb{G}^X$. Then every $\mathbb{G}^X$-local martingale $L = (L_t)_{0 \leq t \leq T}$ admits a representation of the form:*

$$L_t = L_0 + \int_0^t \varphi_u \, \mathrm{d}B_u + \int_0^t \psi_u \, \mathrm{d}M_u^{\mathbb{G}^X}, \qquad \text{for all } t \in [0,T], \tag{3.12}$$

for some $\mathbb{G}^X$-predictable processes $\varphi = (\varphi_t)_{0 \leq t \leq T}$ and $\psi = (\psi_t)_{0 \leq t \leq T}$ with $\int_0^T \varphi_t^2 \, \mathrm{d}t < \infty$ P-a.s. and $\int_0^T |\psi_t| |\mathrm{d}A_t^{\mathbb{G}^X}| < \infty$ P-a.s.

Proof. As in Proposition 3.8, NA1 holds if and only if $\int_0^T (\mu_t/V_t)^2 \, \mathrm{d}t < \infty$ P-a.s. Hence, we can define the strictly positive $\mathbb{G}^X$-local martingale $\widehat{Z} := \mathcal{E}(-\int (\mu/V) \, \mathrm{d}B)$. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be a localizing sequence for $\widehat{Z}$, meaning that $\widehat{Z}^{\tau_n}$ is a uniformly integrable $\mathbb{G}^X$-martingale, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define the filtration $\mathbb{G}^{X,n} := (\mathcal{G}^X_{t \wedge \tau_n})_{0 \leq t \leq T}$ and, analogously, $\mathbb{F}^{\widehat{Y},n} := (\mathcal{F}^{\widehat{Y}}_{t \wedge \tau_n})_{0 \leq t \leq T}$ and $\mathbb{G}^{\widehat{Y},n} := (\mathcal{G}^{\widehat{Y}}_{t \wedge \tau_n})_{0 \leq t \leq T}$. For every $n \in \mathbb{N}$, let the probability measure Q^n be defined on $\mathcal{G}^X_{T \wedge \tau_n}$ by $\frac{dQ^n}{dP} := \widehat{Z}_{T \wedge \tau_n}$. Girsanov's theorem implies that $(M^{\mathbb{G}^X})^{\tau_n}$ is a $(Q^n, \mathbb{G}^{X,n})$-martingale, since $[\widehat{Z}, M^{\mathbb{G}^X}]^{\tau_n} = 0$, for every $n \in \mathbb{N}$. Again by Girsanov's theorem, the stopped process $\widehat{Y}^{\tau_n}$ is also a $(Q^n, \mathbb{G}^{X,n})$-local martingale, for every $n \in \mathbb{N}$. Indeed, for all $t \in [0,T]$:

$$\widehat{Y}_t^{\tau_n} = B_t^{\tau_n} + \int_0^{t \wedge \tau_n} \frac{\mu_u}{V_u} \, \mathrm{d}u = B_t^{\tau_n} - \int_0^{t \wedge \tau_n} \frac{1}{\widehat{Z}_u^{\tau_n}} \, \mathrm{d}[\widehat{Z}, B]_u^{\tau_n}.$$

In particular, since $[\widehat{Y}^{\tau_n}]_t = [B]_t^{\tau_n} = \tau_n \wedge t$, for all $t \in [0,T]$, the process $\widehat{Y}^{\tau_n}$ is a stopped $(Q^n, \mathbb{F}^{\widehat{Y},n})$-Brownian motion. Due to Lemma 13.8 of He et al. [19], the predictable representation property is stable under stopping and, as a consequence, every $(Q^n, \mathbb{F}^{\widehat{Y},n})$-local martingale can be represented as a stochastic integral of $\widehat{Y}^{\tau_n}$. Since $\widehat{Y}^{\tau_n}$ is also a stopped $(Q^n, \mathbb{G}^{X,n})$-Brownian motion, all $(Q^n, \mathbb{F}^{\widehat{Y},n})$-local martingales are also $(Q^n, \mathbb{G}^{X,n})$-local martingales. Hence, Theorem 2.3 of Kusuoka [32] together with Lemma 3.9 implies that any

[3] The $\mathbb{G}^X$-predictable compensator $A^{\mathbb{G}^X}$ of τ is given by the dual $\mathbb{G}^X$-predictable projection of the $\mathbb{G}$-predictable compensator $A^{\mathbb{G}}$ of τ. Indeed, the process $M^{\mathbb{G}} := \mathbf{1}_{\{\tau \leq \cdot\}} - A^{\mathbb{G}}$ is a uniformly integrable $\mathbb{G}$-martingale, due to the Doob-Meyer decomposition. Hence, its $\mathbb{G}^X$-optional projection ${}^oM^{\mathbb{G}}$ is a $\mathbb{G}^X$-martingale, where ${}^oM^{\mathbb{G}} = \mathbf{1}_{\{\tau \leq \cdot\}} - {}^oA^{\mathbb{G}}$, since τ is a $\mathbb{G}^X$-stopping time. Let $A^{\mathbb{G},p}$ be the dual $\mathbb{G}^X$-predictable projection of $A^{\mathbb{G}}$ (which always exists since $A^{\mathbb{G}}$ is integrable), so that ${}^oA^{\mathbb{G}} - A^{\mathbb{G},p} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}^X)$. Writing ${}^oM^{\mathbb{G}} = \mathbf{1}_{\{\tau \leq \cdot\}} - {}^oA^{\mathbb{G}} + A^{\mathbb{G},p} - A^{\mathbb{G},p}$ shows that $A^{\mathbb{G},p}$ is the $\mathbb{G}^X$-predictable compensator of τ. Together with 1.40 of Jacod [21] this implies that, if $A^{\mathbb{G}}$ is absolutely continuous with respect to $\mathrm{d}t$, then also $A^{\mathbb{G}^X}$ will be absolutely continuous with respect to $\mathrm{d}t$. In particular, this means that if τ is a *totally inaccessible* $\mathbb{G}$-stopping time (see He et al. [19], Definition 4.19) with absolutely continuous $\mathbb{G}$-predictable compensator then it remains totally inaccessible with respect to $\mathbb{G}^X$.

$(Q^n, \mathbb{G}^{X,n})$-local martingale $\hat{L} = (\hat{L}_t)_{0\leq t\leq T}$ can be represented as:

$$\hat{L}_t = \hat{L}_0 + \int_0^t \varphi_u \, \mathrm{d}\widehat{Y}_u^{\tau_n} + \int_0^t \psi_u \, \mathrm{d}(M^{\mathbb{G}^X})_u^{\tau_n}, \qquad \text{for all } t \in [0,T], \tag{3.13}$$

where $\varphi = (\varphi_t)_{0\leq t\leq T}$ and $\psi = (\psi_t)_{0\leq t\leq T}$ are two $\mathbb{G}^{X,n}$-predictable processes such that $\int_0^T \varphi_t^2 \, \mathrm{d}t < \infty$ P-a.s. and $\int_0^T |\psi_t||\mathrm{d}A_t^{\mathbb{G}^X}| < \infty$ P-a.s. Let $L = (L_t)_{0\leq t\leq T} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}^X)$. By Girsanov's theorem, the difference $L^{\tau_n} - \int \frac{1}{\widehat{Z}^{\tau_n}} \mathrm{d}[\widehat{Z}, L]^{\tau_n}$ is a $(Q^n, \mathbb{G}^{X,n})$-local martingale, for every $n \in \mathbb{N}$. Hence, by (3.13), there exist two $\mathbb{G}^{X,n}$-predictable processes φ^n and ψ^n such that, for every $n \in \mathbb{N}$ and for all $t \in [0,T]$:

$$\begin{aligned} L_t^{\tau_n} &= L_0 + \int_0^t \varphi_u^n \, \mathrm{d}\widehat{Y}_u^{\tau_n} + \int_0^t \psi_u^n \, \mathrm{d}(M^{\mathbb{G}^X})_u^{\tau_n} + \int_0^t \frac{1}{\widehat{Z}_u^{\tau_n}} \, \mathrm{d}[\widehat{Z}, L]_u^{\tau_n} \\ &= L_0 + \int_0^t \varphi_u^n \, \mathrm{d}B_u^{\tau_n} + \int_0^t \psi_u^n \, \mathrm{d}(M^{\mathbb{G}^X})_u^{\tau_n} + \int_0^t \frac{1}{\widehat{Z}_u^{\tau_n}} \, \mathrm{d}[\widehat{Z}, L]_u^{\tau_n} + \int_0^{t\wedge\tau_n} \varphi_u^n \frac{\mu_u}{V_u} \, \mathrm{d}u. \end{aligned} \tag{3.14}$$

Since the processes L^{τ_n}, $\int \varphi^n \, \mathrm{d}B^{\tau_n}$ and $\int \psi^n \, \mathrm{d}(M^{\mathbb{G}^X})^{\tau_n}$ are all $(P, \mathbb{G}^X)$-local martingales, the finite variation term in (3.14) must vanish and thus:

$$L_t^{\tau_n} = L_0 + \int_0^t \varphi_u^n \, \mathrm{d}B_u^{\tau_n} + \int_0^t \psi_u^n \, \mathrm{d}(M^{\mathbb{G}^X})_u^{\tau_n},$$

for every $n \in \mathbb{N}$ and for all $t \in [0,T]$. The representation (3.12) follows by letting the $\mathbb{G}^X$-predictable processes $\varphi = (\varphi_t)_{0\leq t\leq T}$ and $\psi = (\psi_t)_{0\leq t\leq T}$ be defined as:

$$\varphi = \sum_{n=1}^{\infty} \mathbf{1}_{(\!(\tau_{n-1},\tau_n]\!]} \varphi^n \qquad \text{and} \qquad \psi = \sum_{n=1}^{\infty} \mathbf{1}_{(\!(\tau_{n-1},\tau_n]\!]} \psi^n.$$

□

In particular, Proposition 3.10 allows us to obtain an explicit description of the family of all equivalent local martingale measures for the model (2.1)-(2.3) when considered with respect to the filtration $\mathbb{G}^X$. The following corollary is an immediate consequence of Proposition 3.8 and Proposition 3.10, noting that $[B, M^{\mathbb{G}^X}] = 0$.

Corollary 3.11. *Suppose that Conditions I and II hold. Then NFLVR holds for the model* (2.1)-(2.3) *in the filtration* $\mathbb{G}^X$ *if and only if NA1 holds and* $E\big[\mathcal{E}(-\int (\mu/V)\mathrm{d}B + \int \psi \, \mathrm{d}M^{\mathbb{G}^X})_T\big] = 1$ *for some* $\mathbb{G}^X$*-predictable process* ψ *such that* $\psi \Delta M^{\mathbb{G}^X} > -1$ *and* $\int_0^T |\psi_t||\mathrm{d}A_t^{\mathbb{G}^X}| < \infty$ *P-a.s.*

3.2.3. *Stability of no-arbitrage conditions with respect to filtration shrinkage.* At this point, one may wonder whether the absence of arbitrage in $\mathbb{G}$ already implies the absence of arbitrage in the smaller filtration $\mathbb{G}^X$. This question is also related to the behavior of local martingales with respect to a filtration shrinkage, as in Föllmer and Protter [12]. Intuitively, the answer to such a question is expected to be affirmative, because any outcome of a $\mathbb{G}^X$-trading strategy can be realized as an outcome of a $\mathbb{G}$-trading strategy, since $\mathbb{G}^X \subset \mathbb{G}$.

Proposition 3.12. *Suppose that Conditions I and II hold. Then the following assertions hold for the model* (2.1)-(2.3)*:*

(a) *NA1 in the filtration* $\mathbb{G}$ *implies NA1 in the filtration* $\mathbb{G}^X$*;*
(b) *NFLVR in the filtration* $\mathbb{G}$ *implies NFLVR in the filtration* $\mathbb{G}^X$*.*

Proof. If NA1 holds in $\mathbb{G}$, then Proposition 3.3 shows that $\int_0^T \bar{\theta}_t^2 \,\mathrm{d}t < \infty$ P-a.s. or, equivalently, $\int_0^T (\tilde{\mu}_t/V_t)^2 \,\mathrm{d}t < \infty$ P-a.s., using the notation introduced in (3.3)-(3.6). We now show that this implies that $L(S, \mathbb{G}^X) \subset L(S, \mathbb{G})$. The claim will then follow directly from Definition 2.2. Let $h \in L(S, \mathbb{G}^X)$. Since S is continuous and due to equation (3.9), this implies that $hS \in L^2_{\mathrm{loc}}\big(\int V \mathrm{d}B, \mathbb{G}^X\big) \subset L^2_{\mathrm{loc}}\big(\int V \mathrm{d}\widetilde{W}, \mathbb{G}\big)$, i.e., $\int_0^T (h_t S_t V_t)^2 \mathrm{d}t < \infty$ P-a.s. Due to the Cauchy-Schwarz inequality:

$$\int_0^T |h_t S_t \tilde{\mu}_t| \,\mathrm{d}t = \int_0^T \Big| h_t S_t V_t \frac{\tilde{\mu}_t}{V_t} \Big| \mathrm{d}t \leq \left(\int_0^T (h_t S_t V_t)^2 \mathrm{d}t \right)^{1/2} \left(\int_0^T \left(\frac{\tilde{\mu}_t}{V_t} \right)^2 \mathrm{d}t \right)^{1/2} < \infty \qquad P\text{-a.s.}$$

We have thus shown that $hS \in L\big(\int_0^{\cdot} \tilde{\mu}_u \,\mathrm{d}u, \mathbb{G}\big) \cap L^2_{\mathrm{loc}}\big(\int V \mathrm{d}\widetilde{W}, \mathbb{G}\big) = L(X, \mathbb{G})$, using equation (3.2). Hence, we can conclude that $h \in L(S, \mathbb{G})$. □

3.3. **Special cases.** In this section we analyze two special (and important, especially in view of financial applications) cases of the general setting described so far. In particular, Section 3.3.1 deals with the case where the random time τ is assumed to satisfy the *immersion property* between the filtrations $\mathbb{F}$ and $\mathbb{G}$, meaning that all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales. Section 3.3.2 specializes the results obtained in the preceding sections for the simple but interesting case where the random time τ is a stopping time for the filtration $\mathbb{F}$.

3.3.1. *Immersion property between* $\mathbb{F}$ *and* $\mathbb{G}$*.* Let us suppose that the filtrations $\mathbb{F}$ and $\mathbb{G}$ satisfy the *immersion property* (or $(\mathcal{H})$*-hypothesis*, see Brémaud and Yor [3]) with respect to the random time τ, meaning that all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales. This situation is rather interesting in view of the fact that many random time models considered in financial and insurance mathematics satisfy this property. For instance, in credit risk modeling, τ typically represents a default event and is often assumed to be a *doubly stochastic random time* with a given $\mathbb{F}$-adapted default intensity (in particular, τ can be constructed via the canonical approach presented in Bielecki and Rutkowski [1], Section 8.2.1). In this case, the immersion property holds between $\mathbb{F}$ and $\mathbb{G}$ and τ is a random change point that occurs in an unpredictable way.

If the immersion property holds between $\mathbb{F}$ and $\mathbb{G}$, then the analysis of the model (2.1)-(2.3) simplifies considerably. Indeed, if all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, Condition II trivially holds with $\theta^i \equiv 0$ for $i = 1, 2$. In this case, if Condition I holds, Proposition 3.3 immediately implies that the model (2.1)-(2.3) does not allow for arbitrages of the first kind. However, as can be deduced from Corollary 3.3, we cannot a priori exclude the existence of free lunches with vanishing risk.

Furthermore, if $\mathbb{F} = \mathbb{F}^{W^1} \vee \mathbb{F}^{W^2}$ and $\mathbb{F}$ is immersed in $\mathbb{G}$, Theorem 2.3 of Kusuoka [32] provides the martingale representation (3.7) in the filtration $\mathbb{G}$. Hence, by combining part (c) of Proposition 3.3 with (3.8) we get a complete description of the family of all equivalent local martingale measures for the model (2.1)-(2.3) considered in the filtration $\mathbb{G}$.

Since τ is a $\mathbb{G}^X$-stopping time, it is easy to deduce from (3.2)-(3.5) and Lemma 3.6 that the canonical decomposition of X in the filtration $\mathbb{G}^X$ coincides with its canonical decomposition in $\mathbb{G}$. Furthermore, given that $\bar{\theta} \equiv 0$ (and hence ${}^p\bar{\theta} \equiv 0$), Proposition 3.8 implies NA1 for the model (2.1)-(2.3) considered in the filtration $\mathbb{G}^X$.

3.3.2. *Stopping times with respect to the filtration* $\mathbb{F}$. Let us now consider the case where τ is a stopping time with respect to the filtration $\mathbb{F}$. For instance, τ could be defined as the first passage time of one of the two Brownian motions W^1 and W^2 at some given level. This is the case where τ is a random change point which is *endogenous* to the model, in the sense that its occurrence is determined by the same stochastic processes which drive the dynamics of the asset price process S.

If τ is an $\mathbb{F}$-stopping time, it is evident that $\mathbb{G} = \mathbb{F}$. Hence, Condition II is trivially satisfied with $\theta^i \equiv 0$ for $i = 1, 2$. In this case, as discussed in Section 3.3.1, Proposition 3.3 implies that NA1 holds in $\mathbb{F} = \mathbb{G}$. However, we cannot exclude a priori the existence of free lunches with vanishing risk. When considering the model (2.1)-(2.3) from the point of view of the filtration $\mathbb{G}^X$, we are in a situation analogous to that discussed at the end of Section 3.3.1.

4. The price process filtration $\mathbb{F}^X$

In this section, we study the model (2.1)-(2.3) with respect to its own filtration $\mathbb{F}^X$, which is the smallest among all filtrations considered in Section 2.

4.1. **Canonical decomposition and no-arbitrage properties in $\mathbb{F}^X$.** In general, the random time τ is not necessarily an $\mathbb{F}^X$-stopping time. Nevertheless, the process $V = (V_t)_{0\le t\le T}$ introduced in (3.4) is $\mathbb{F}^X$-predictable, as shown in the following lemma.

Lemma 4.1. *Suppose that Conditions I and II hold. Then the process $V = (V_t)_{0\le t\le T}$ introduced in* (3.4) *is $\mathbb{F}^X$-predictable.*

Proof. For every $\omega \in \Omega$ and any $t \in [0, T]$, because $[X]_t = \int_0^t V_u^2 \,\mathrm{d}u$, the quadratic variation is differentiable with respect to t and the derivative is

$$\frac{\partial}{\partial t}[X]_t(\omega) = V_t^2(\omega) = \mathbf{1}_{\{t\le\tau(\omega)\}}\big(\sigma^1\big(t, X_t(\omega)\big)\big)^2 + \mathbf{1}_{\{t>\tau(\omega)\}}\big(\sigma^2\big(t, X_t(\omega)\big)\big)^2.$$

For all $t \in (0, T]$, the derivative $\frac{\partial}{\partial t}[X]_t$ is $\mathcal{F}_t^X$-measurable, because it equals the left derivative $\lim_{\epsilon\searrow 0}\big([X]_t - [X]_{t-\epsilon}\big)/\epsilon$, while for $t = 0$ we have $V_0^2 = \big(\sigma^1(0,0)\big)^2$. This implies that the process $V = (V_t)_{0\le t\le T}$ is $\mathbb{F}^X$-adapted. Being left-continuous, due to part (b) of Condition I, it is also $\mathbb{F}^X$-predictable. □

The next lemma gives the canonical decomposition of X in its own filtration $\mathbb{F}^X$.

Lemma 4.2. *Suppose that Conditions I and II hold. Then the process X admits the following canonical decomposition with respect to the filtration $\mathbb{F}^X$:*

$$X_t = \int_0^t \bar{\mu}_u \,\mathrm{d}u + \int_0^t V_u \,\mathrm{d}\bar{B}_u, \qquad \textit{for all } t \in [0, T], \tag{4.1}$$

where the $\mathbb{F}^X$-predictable process $\bar{\mu} = (\bar{\mu}_t)_{0\le t\le T}$ is defined as follows, for all $t \in [0, T]$:

$$\begin{aligned}\bar{\mu}_t := {}& {}^p(\mathbf{1}_{[\![0,\tau]\!]})_t\, \mu^1(t, X_t) + {}^p\big(\mathbf{1}_{[\![0,\tau]\!]}\theta^1\big)_t\, \sigma^1(t, X_t) \\ &+ {}^p(\mathbf{1}_{(\!(\tau,T]\!]})_t\, \mu^2(t, X_t) + {}^p\Big(\mathbf{1}_{(\!(\tau,T]\!]}\big(\rho\,\theta^1 + \sqrt{1-\rho^2}\,\theta^2\big)\Big)_t\, \sigma^2(t, X_t),\end{aligned} \tag{4.2}$$

with ^{p} denoting the $\mathbb{F}^X$-predictable projection and where the process $\bar{B} = (\bar{B}_t)_{0\le t\le T}$ is an $\mathbb{F}^X$-Brownian motion and $V = (V_t)_{0\le t\le T}$ is as in (3.4).

Proof. The proof is similar to the proof of Lemma 3.6. Let $X = \bar{A} + \bar{M}$ be the canonical decomposition of X in $\mathbb{F}^X$, where $\bar{A}$ is a continuous $\mathbb{F}^X$-predictable process of finite variation with $\bar{A}_0 = 0$ and $\bar{M} \in \mathcal{M}_{\rm loc}(\mathbb{F}^X)$ with $\bar{M}_0 = 0$. For every $n \in \mathbb{N}$, define the $\mathbb{F}^X$-stopping time $\tau_n := \inf\{t \in [0,T] : |X_t| \geq n\} \wedge T$. Clearly, we have $\tau_n \nearrow T$, P-a.s. as $n \to \infty$. As in the proof of Lemma 3.6, the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is an $\mathbb{F}^X$-localizing sequence for the $\mathbb{F}^X$-local martingale $\bar{M}$ as well as for the $\mathbb{G}$-local martingale $\int V \mathrm{d}\widetilde{W}$. Proposition 9.24 of Jacod [21] together with (3.2) then implies that $\bar{A} = \left(\int_0^\cdot \tilde{\mu}_t \,\mathrm{d}t\right)^p = \int_0^\cdot {}^p\tilde{\mu}_t \,\mathrm{d}t$. To complete the proof, Lemma 4.1 shows that the process V is $\mathbb{F}^X$-predictable. Hence, the stochastic integral $\bar{B} := \int V^{-1} \mathrm{d}\bar{M}$ is a continuous $\mathbb{F}^X$-local martingale with $\bar{B}_0 = 0$. Since $[\bar{B}]_t = t$, for all $t \in [0,T]$, we conclude that $\bar{B}$ is an $\mathbb{F}^X$-Brownian motion. □

The next proposition answers the question of whether the model (2.1)-(2.3), considered now with respect to its own filtration $\mathbb{F}^X$, admits arbitrage possibilities. The proof is similar to that of Proposition 3.3 and, hence, omitted.

Proposition 4.3. *Suppose that Conditions I and II hold. Then the following assertions hold for the model* (2.1)-(2.3) *considered with respect to the filtration* $\mathbb{F}^X$*:*

(a) *NA1 holds if and only if* $\int_0^T (\bar{\mu}_t/V_t)^2 \,\mathrm{d}t < \infty$ *P-a.s.*
(b) *NFLVR holds if and only if NA1 holds and there exists* $N = (N_t)_{0\leq t\leq T} \in \mathcal{M}_{\rm loc}(\mathbb{F}^X)$ *with* $N_0 = 0$, $\Delta N > -1$ *P-a.s.,* $[N, \bar{B}] = 0$*, such that* $E\left[\mathcal{E}\left(-\int(\bar{\mu}/V)\mathrm{d}\bar{B} + N\right)_T\right] = 1$.

4.2. **The $\mathbb{F}^X$-martingale representation property.** This section will provide the martingale representation property with respect to the price filtration $\mathbb{F}^X$. We shall see that the representation formula and thus the market completeness or incompleteness are determined by the relationship of the two volatility functions σ^1 and σ^2, which, by the continuity assumption of Condition I (b), falls into one and only one of the three situations below.

Condition III. (a) Distinct volatility functions

$$\sigma^1(t,x) \neq \sigma^2(t,x), \text{ for any } (t,x) \in [0,T] \times \mathbb{R}.$$

(b) Identical volatilities

$$\sigma^1(t,x) = \sigma^2(t,x) =: \sigma(t,x), \text{ for all } (t,x) \in [0,T] \times \mathbb{R}.$$

(c) The volatilities differ on an open set

$$\mathcal{O} := \left\{(t,x) \in [0,T] \times \mathbb{R} : \sigma^1(t,x) \neq \sigma^2(t,x)\right\},$$

which is neither the empty set nor the whole space $[0,T] \times \mathbb{R}$.

We provide the representation theorems respectively under Condition III-(a) and (b). Then to get the predictable martingale representation under Condition III-(c), it suffices to apply the results in the other two situations according to whether the two volatilities evaluated at the current value of X are equal or not, as we are now going to show.

When Condition III-(c) holds, without loss of generality, we may assume that $\sigma^1(0,0) \neq \sigma^2(0,0)$. Let $\rho_0 = 0$. For $k = 1, 2, \cdots$, a non-decreasing sequence of $\mathbb{F}^X$-stopping times $\{\rho_k\}_{k=1}^\infty$ is iteratively defined as

$$\begin{aligned} \rho_{2k-1} &:= \inf\left\{\rho_{2k-2} < t \leq T : (t, X_t) \notin \mathcal{O}\right\} \wedge T; \\ \rho_{2k} &:= \inf\left\{\rho_{2k-1} < t \leq T : (t, X_t) \in \mathcal{O}\right\} \wedge T. \end{aligned} \tag{4.3}$$

Since $\mathcal{O}$ is an open set and the process X is continuous, the two volatilities $\sigma^1(t, X_t)$ and $\sigma^2(t, X_t)$ equal on $\bigcup_{k=1}^{\infty} [\![\rho_{2k-1}, \rho_{2k}]\!]$, and do not equal on $\bigcup_{k=1}^{\infty} (\!(\rho_{2k-2}, \rho_{2k-1})\!)$. Because the martingale representation is a local property, an $\mathbb{F}^X$-martingale is represented as in Proposition 3.10 over the stochastic intervals $(\!(\rho_{2k-2}, \rho_{2k-1}]\!]$, and as in Proposition 4.6 over the stochastic intervals $(\!(\rho_{2k-1}, \rho_{2k}]\!]$. The intervals are chosen to be open at the left end point and closed at the right end point in order to preserve the predictability of the coefficients in the representation formulae.

4.2.1. *Distinct volatility functions.* Let us first analyze the case where σ^1 and σ^2 differ everywhere. Condition III-(a) means that the volatility of the process X is entirely different immediately after the random time τ. Under Condition III-(a), the result below explicitly characterizes the price filtration $\mathbb{F}^X$.

Proposition 4.4. *Suppose that Conditions I, II and III-(a) hold. Then τ is a stopping time for the filtration $\mathbb{F}^X$ and, consequently, the filtrations $\mathbb{F}^X$ and $\mathbb{G}^X$ coincide.*

Proof. Condition III-(a) and the proof of Lemma 4.1 together imply that, for any $t \in [0, T]$:

$$\{\omega \in \Omega : \tau(\omega) \geq t\} = \left\{\omega \in \Omega : \frac{\partial}{\partial t}[X]_t(\omega) = \left(\sigma^1\big(t, X_t(\omega)\big)\right)^2\right\} \in \mathcal{F}_t^X.$$

Together with the right continuity of the filtration $\mathbb{F}^X$, this implies that $\{\tau \leq t\} \in \mathcal{F}_t^X$ for all $t \in [0, T]$, hence by definition τ is an $\mathbb{F}^X$-stopping time. □

Due to Proposition 4.4, as long as Condition III-(a) holds, all results obtained in Section 3.2 for the progressively enlarged filtration $\mathbb{G}^X$ are true also for the filtration $\mathbb{F}^X$. In particular, Lemma 3.9 gives an explicit description of the filtration $\mathbb{F}^X = \mathbb{G}^X$ as the progressive enlargement of the filtration $\mathbb{F}^{\widehat{Y}}$ generated by the drifted $\mathbb{F}^X$-Brownian motion $\widehat{Y}$. Furthermore, any $\mathbb{F}^X$-local martingale admits the representation obtained in Proposition 3.10. Summing up, as long as Condition III-(a) holds, our results provide a complete and explicit description of the filtration $\mathbb{F}^X$ generated by the process X.

4.2.2. *Identical volatilities.* Let us now analyze the case of Condition III-(b), where the two volatility functions σ^1 and σ^2 coincide everywhere. This is the case when the volatility of the asset price model described by the SDE (2.2) does not change at the random time τ. Lemma 4.2 gives the following canonical decomposition of X in its own filtration $\mathbb{F}^X$:

$$X_t = \int_0^t \bar{\mu}_u \, \mathrm{d}u + \int_0^t \sigma(u, X_u) \, \mathrm{d}\bar{B}_u, \qquad \text{for all } t \in [0, T], \tag{4.4}$$

where the process $\bar{\mu} = (\bar{\mu}_t)_{0 \leq t \leq T}$ is as in (4.2). Since the continuous process $\sigma(\cdot, X_\cdot)$ is $\mathbb{F}^X$-adapted and never attains zero, it is $\mathbb{F}^X$-predictable and bounded away from zero. Hence, we can define a drifted $\mathbb{F}^X$-Brownian motion $\bar{Y} = (\bar{Y}_t)_{0 \leq t \leq T}$ as follows, for all $t \in [0, T]$:

$$\bar{Y}_t := \int_0^t \frac{1}{\sigma(u, X_u)} \, \mathrm{d}X_u = \int_0^t \frac{\bar{\mu}_u}{\sigma(u, X_u)} \, \mathrm{d}u + \bar{B}_t. \tag{4.5}$$

Denote by $\mathbb{F}^{\bar{Y}} = (\mathcal{F}_t^{\bar{Y}})_{0 \leq t \leq T}$ its right-continuous P-augmented natural filtration. We can now easily prove the next lemma, which shows that $\mathbb{F}^X$ coincides with the filtration $\mathbb{F}^{\bar{Y}}$ generated by the drifted Brownian motion $\bar{Y}$ (see also Section 3 of Pham and Quenez [35] for related results).

Lemma 4.5. *Suppose that Conditions I, II and III-(b) hold. Then* $\mathbb{F}^X = \mathbb{F}^{\bar{Y}}$.

Proof. Clearly, the process $\bar{Y}$ defined in (4.5) is $\mathbb{F}^X$-adapted and, hence, we have $\mathbb{F}^{\bar{Y}} \subseteq \mathbb{F}^X$. To prove the converse inclusion, it suffices to note that the process X can be represented as $X_t = \int_0^t \sigma(u, X_u)\,\mathrm{d}\bar{Y}_u$ for all $t \in [0,T]$. The same arguments used in the proof of Lemma 3.9 allow then to show that $\mathbb{F}^X \subseteq \mathbb{F}^{\bar{Y}}$, noting that the function σ is deterministic. □

As in Section 3.2, we can now prove a martingale representation result for the filtration $\mathbb{F}^X$ in the special case where the two volatility functions σ^1 and σ^2 coincide.

Proposition 4.6. *Suppose that Conditions I, II and III-(b) hold and assume in addition that NA1 holds in the filtration $\mathbb{F}^X$. Then every $\mathbb{F}^X$-local martingale $L = (L_t)_{0\leq t\leq T}$ admits a representation of the form:*

$$L_t = L_0 + \int_0^t \varphi_u\,\mathrm{d}\bar{B}_u, \qquad \textit{for all } t \in [0,T],$$

for some $\mathbb{F}^X$-predictable process $\varphi = (\varphi_t)_{0\leq t\leq T}$ with $\int_0^T \varphi_t^2\,\mathrm{d}t < \infty$ P-a.s.

Proof. The proof is similar to the proof of Proposition 3.10. As shown in Proposition 4.3, NA1 holds if and only if $\int_0^T \big(\bar{\mu}_t/\sigma(t,X_t)\big)^2 \mathrm{d}t < \infty$ P-a.s. Hence, we can define the strictly positive $\mathbb{F}^X$-local martingale $\widehat{Z} := \mathcal{E}(-\int \frac{\bar{\mu}}{\sigma(\cdot,X)}\,\mathrm{d}\bar{B})$. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a localizing sequence for $\widehat{Z}$, meaning that $\widehat{Z}^{\tau_n}$ is a uniformly integrable $\mathbb{F}^X$-martingale, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define the filtrations $\mathbb{F}^{X,n} := (\mathcal{F}^X_{t\wedge\tau_n})_{0\leq t\leq T}$ and $\mathbb{F}^{\bar{Y},n} := (\mathcal{F}^{\bar{Y}}_{t\wedge\tau_n})_{0\leq t\leq T}$ and define the probability measure Q^n by letting $\frac{dQ^n}{dP} := \widehat{Z}_{T\wedge\tau_n}$. An application of Girsanov's theorem shows that the stopped process $\bar{Y}^{\tau_n}$ is a $(Q^n, \mathbb{F}^{X,n})$-local martingale, for every $n \in \mathbb{N}$. More precisely, the process $\bar{Y}^{\tau_n}$ is a stopped $(Q^n, \mathbb{F}^{\bar{Y},n})$-Brownian motion. As a consequence, using Lemma 4.5, every $(Q^n, \mathbb{F}^{X,n})$-local martingale $\hat{L} = (\hat{L}_t)_{0\leq t\leq T}$ can be represented as:

$$\hat{L}_t = \hat{L}_0 + \int_0^t \varphi_u\,\mathrm{d}\bar{Y}^{\tau_n}_u, \qquad \text{for all } t \in [0,T],$$

where $\varphi = (\varphi_t)_{0\leq t\leq T}$ is an $\mathbb{F}^{X,n}$-predictable process such that $\int_0^T \varphi_t^2\,\mathrm{d}t < \infty$ P-a.s. Let $L = (L_t)_{0\leq t\leq T} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{F}^X)$. By Girsanov's theorem, the difference $L^{\tau_n} - \int \frac{1}{\widehat{Z}^{\tau_n}}\,\mathrm{d}[\widehat{Z}, L]^{\tau_n}$ is a $(Q^n, \mathbb{F}^{X,n})$-local martingale, for every $n \in \mathbb{N}$. Hence, there exists an $\mathbb{F}^{X,n}$-predictable process φ^n such that the following hold, for every $n \in \mathbb{N}$ and for all $t \in [0,T]$:

$$\begin{aligned} L^{\tau_n}_t &= L_0 + \int_0^t \varphi^n_u\,\mathrm{d}\bar{Y}^{\tau_n}_u + \int_0^t \frac{1}{\widehat{Z}^{\tau_n}_u}\,\mathrm{d}[\widehat{Z}, L]^{\tau_n}_u \\ &= L_0 + \int_0^t \varphi^n_u\,\mathrm{d}\bar{B}^{\tau_n}_u + \int_0^t \frac{1}{\widehat{Z}^{\tau_n}_u}\,\mathrm{d}[\widehat{Z}, L]^{\tau_n}_u + \int_0^{t\wedge\tau_n} \varphi^n_u \frac{\bar{\mu}_u}{\sigma(u,X_u)}\,\mathrm{d}u. \end{aligned}$$

Since $L^{\tau_n}, \int \varphi^n\,\mathrm{d}\bar{B}^{\tau_n} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{F}^X)$, the finite variation term must vanish and thus

$$L^{\tau_n}_t = L_0 + \int_0^t \varphi^n_u\,\mathrm{d}\bar{B}^{\tau_n}_u, \qquad \text{for all } t \in [0,T] \text{ and for every } n \in \mathbb{N}.$$

The claim then follows by letting $\varphi = \sum_{n=1}^\infty \mathbf{1}_{(\!(\tau_{n-1},\tau_n]\!]}\varphi^n$. □

By relying on Proposition 4.6 we can prove that the financial market where the asset S is traded with respect to the information contained in its own filtration $\mathbb{F}^X$ is *complete*, in the sense that all bounded $\mathcal{F}_T^X$-measurable contingent claims can be perfectly replicated.

Corollary 4.7. *Under the same assumptions of Proposition 4.6, the financial market $(S, \mathbb{F}^X)$ is complete, i.e., for any bounded $\mathcal{F}_T^X$-measurable non-negative random variable H there exists a couple $(v^H, h^H) \in [0,\infty) \times L(S, \mathbb{F}^X)$ such that $H = v^H + \int_0^T h_t^H \,\mathrm{d}S_t$ P-a.s.*

Proof (outline). The claim follows by relying on the same arguments of the proof of Theorem 4.5.5 of Fontana and Runggaldier [14] together with equation (4.4) and Proposition 4.6. □

Note that Corollary 4.7 only requires the NA1 no-arbitrage condition for the model (2.1)-(2.3) in its own filtration $\mathbb{F}^X$ and does not depend on the validity of NFLVR.

5. The initially enlarged filtrations $\mathbb{G}^{(\tau)}$ and $\mathbb{G}^{X,(\tau)}$

In the remainder of the paper we turn our attention to the study of the model (2.1)-(2.3) in the initially enlarged filtrations $\mathbb{G}^{(\tau)}$ and $\mathbb{G}^{X,(\tau)}$. Recall that the filtration $\mathbb{G}^{(\tau)} = (\mathcal{G}_t^{(\tau)})_{0\le t\le T}$ is defined by $\mathcal{G}_t^{(\tau)} := \bigcap_{s>t}\big(\mathcal{F}_s \vee \sigma(\tau)\big)$ and the filtration $\mathbb{G}^{X,(\tau)} = (\mathcal{G}_t^{X,(\tau)})_{0\le t\le T}$ by $\mathcal{G}_t^{X,(\tau)} := \bigcap_{s>t}\big(\mathcal{F}_s^X \vee \sigma(\tau)\big)$, for all $t \in [0,T]$.

5.1. The initially enlarged filtration $\mathbb{G}^{(\tau)}$.

We begin the analysis of the model (2.1)-(2.3) by assuming that the model is well-defined in the enlarged filtration $\mathbb{G}^{(\tau)}$. More precisely, the condition imposed below ensures that the driving Brownian motions W^1 and W^2 remain semimartingales with respect to the filtration $\mathbb{G}^{(\tau)}$.

Condition IV. There exist two $\mathbb{G}^{(\tau)}$-predictable processes $\theta^{1,(\tau)} = (\theta_t^{1,(\tau)})_{0\le t\le T}$ and $\theta^{2,(\tau)} = (\theta_t^{2,(\tau)})_{0\le t\le T}$ and two $\mathbb{G}^{(\tau)}$-Brownian motions $W^{1,(\tau)} = (W_t^{1,(\tau)})_{0\le t\le T}$ and $W^{2,(\tau)} = (W_t^{2,(\tau)})_{0\le t\le T}$ such that for all $t \in [0,T]$:

$$W_t^i = W_t^{i,(\tau)} + \int_0^t \theta_u^{i,(\tau)} \,\mathrm{d}u, \qquad \text{for } i = 1,2. \tag{5.1}$$

Remarks 5.1. 1) We emphasize that here the superscript $^{(\tau)}$ is used to denote processes that are adapted to the filtration $\mathbb{G}^{(\tau)}$ and should not be confused with the superscript $^{\tau}$ used earlier for processes stopped at time τ. For example, $Y^{(\tau)} = (Y_t^{(\tau)})_{0\le t\le T}$ is a process adapted to the filtration $\mathbb{G}^{(\tau)}$ and Y^τ denotes the process Y stopped at τ, i.e. $Y^\tau = (Y_{t\wedge\tau})_{0\le t\le T}$.

2) Note that the $\mathbb{G}^{(\tau)}$-Brownian motions $W^{i,(\tau)}$, for $i = 1,2$, are independent of the random time τ. This follows directly from the definition of the $\mathbb{G}^{(\tau)}$-Brownian motion, which implies that $W_t^{i,(\tau)} = W_t^{i,(\tau)} - W_0^{i,(\tau)}$ is independent of $\mathcal{G}_0^{(\tau)}$, for every $t \in [0,T]$, and the inclusion $\sigma(\tau) \subseteq \mathcal{G}_0^{(\tau)}$.

Condition IV is satisfied under the classical hypothesis due to Jacod [22], which is typically used in the literature when dealing with initial enlargements of filtrations and which assumes that the (regular) $\mathcal{F}_t$-conditional law of τ admits a density with respect to the unconditional law of τ, for all $t \in [0,T]$. If this condition is verified, we shall say that the *density hypothesis* holds. In particular, if τ is independent of $\mathbb{F}$, the density hypothesis is trivially satisfied (with the constant density 1). In the latter case, condition (5.1) obviously holds with $\theta^{i,(\tau)} \equiv 0$, for $i = 1,2$. In the case where τ is constructed via the canonical construction, a connection

between the $\mathbb{F}$-intensity and the density can be established (see El Karoui et al. [7, Section 4.2]). On the other side, the density hypothesis is *not* valid when τ is an honest time with respect to $\mathbb{F}$. For instance, the case of a stopping time with respect to $\mathbb{F}$ is thus excluded, e.g. the case when τ is the first passage time of one of the two Brownian motions W^1 and W^2 at some given boundary (see Section 3.3.2). In this case a method developed by M. Yor can be applied; see Yor [42, Chapter 12] for a concise description of the method. An example of such an initial enlargement is treated on page 53 of Jeulin [28] and in Jeanblanc and Leniec [24].

Under Condition IV, the process $X = (X_t)_{0\leq t\leq T}$ admits the following canonical decomposition with respect to the filtration $\mathbb{G}^{(\tau)}$ (compare with (3.2)):

$$X_t = \int_0^t \tilde{\mu}_u^{(\tau)}\,\mathrm{d}u + \int_0^t V_u\,\mathrm{d}W_u^{(\tau)}, \qquad \text{for all } t \in [0,T], \tag{5.2}$$

where the processes $\tilde{\mu}^{(\tau)} = (\tilde{\mu}_t^{(\tau)})_{0\leq t\leq T}$ and $W^{(\tau)} = (W_t^{(\tau)})_{0\leq t\leq T}$ are defined as follows, for all $t \in [0,T]$:

$$\begin{aligned}\tilde{\mu}_t^{(\tau)} &:= \mathbf{1}_{\{t\leq\tau\}}\Big(\mu^1(t,X_t) + \sigma^1(t,X_t)\,\theta_t^{1,(\tau)}\Big)\\ &\quad+\mathbf{1}_{\{t>\tau\}}\Big(\mu^2(t,X_t) + \sigma^2(t,X_t)\big(\rho\,\theta_t^{1,(\tau)} + \sqrt{1-\rho^2}\,\theta_t^{2,(\tau)}\big)\Big)\end{aligned} \tag{5.3}$$

$$\begin{aligned}W_t^{(\tau)} &:= \mathbf{1}_{\{t\leq\tau\}}W_t^{1,(\tau)} + \mathbf{1}_{\{t>\tau\}}\Big(\rho W_t^{1,(\tau)} + (1-\rho)W_\tau^{1,(\tau)} + \sqrt{1-\rho^2}\,\big(W_t^{2,(\tau)} - W_\tau^{2,(\tau)}\big)\Big)\\ &= \int_0^{t\wedge\tau}\mathrm{d}W_u^{1,(\tau)} + \rho\int_\tau^{t\vee\tau}\mathrm{d}W_u^{1,(\tau)} + \sqrt{1-\rho^2}\int_\tau^{t\vee\tau}\mathrm{d}W_u^{2,(\tau)},\end{aligned} \tag{5.4}$$

and $V = (V_t)_{0\leq t\leq T}$ is defined in (3.4). By computing its quadratic variation one can easily verify that the process $W^{(\tau)}$ is a $\mathbb{G}^{(\tau)}$-Brownian motion, which is again independent of τ (see Remark 5.1-2). Now we are ready to establish the no-arbitrage properties of the model (2.1)-(2.3) considered in the initially enlarged filtration $\mathbb{G}^{(\tau)}$. We define the $\mathbb{G}^{(\tau)}$-predictable process $\theta^{(\tau)} = (\theta_t^{(\tau)})_{0\leq t\leq T}$, for all $t \in [0,T]$:

$$\theta^{(\tau)} := \mathbf{1}_{[\![0,\tau]\!]}\theta^{1,(\tau)} + \mathbf{1}_{(\!(\tau,T]\!]}\big(\rho\,\theta^{1,(\tau)} + \sqrt{1-\rho^2}\,\theta^{2,(\tau)}\big). \tag{5.5}$$

Proposition 5.2. *Suppose that Conditions I and IV hold. Then the following assertions hold for the model* (2.1)-(2.3) *considered with respect to the filtration* $\mathbb{G}^\tau$*:*

(a) *NA1 holds if and only if* $\int_0^T(\theta_t^{(\tau)})^2\,\mathrm{d}t < \infty$ *P-a.s., with the latter condition being equivalent to* $\int_0^T(\tilde{\mu}_t^{(\tau)}/V_t)^2\,\mathrm{d}t < \infty$ *P-a.s.;*

(b) *NFLVR holds if and only if NA1 holds and there exists some* $N^{(\tau)} = (N_t^{(\tau)})_{0\leq t\leq T} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}^{(\tau)})$ *with* $N_0^{(\tau)} = 0$, $\Delta N^{(\tau)} > -1$ *P-a.s. and* $[N^{(\tau)}, W^{(\tau)}] = 0$ *satisfying* $E[\mathcal{E}(-\int(\tilde{\mu}^{(\tau)}/V)\,\mathrm{d}W^{(\tau)} + N^{(\tau)})_T] = 1$.

Proof. For part (a), by relying on the same arguments used in the proof of Proposition 3.3, it suffices to note that $\int_0^T(\tilde{\mu}_t^{(\tau)}/V_t)^2\,\mathrm{d}t < \infty$ P-a.s. if and only if $\int_0^T(\theta_t^{(\tau)})^2\,\mathrm{d}t < \infty$ P-a.s. In order to prove (b), recall from the proof of Proposition 3.3 (b) that NFLVR holds in $\mathbb{G}^\tau$ if and only if there exists a martingale deflator Z for X with $E[Z_T] = 1$. Part (b) now follows from (5.2) and Lemma 4.3.15 in Fontana and Runggaldier [14]. □

Remark 5.3 (*The $\mathbb{G}^{(\tau)}$-martingale representation property*)**.** Suppose that $\mathbb{F}=\mathbb{F}^{W^1}\vee\mathbb{F}^{W^2}$ and that the density hypothesis holds with a P-a.s. strictly positive density. Then, Proposition 5.3(i) in Callegaro et al. [4] implies that every $\mathbb{G}^{(\tau)}$-local martingale $L^{(\tau)}=(L_t^{(\tau)})_{0\le t\le T}$ admits the representation:

$$L_t^{(\tau)}=L_0^{(\tau)}+\int_0^t \phi_s^{1,(\tau)}\,\mathrm{d}W_s^{1,(\tau)}+\int_0^t \phi_s^{2,(\tau)}\,\mathrm{d}W_s^{2,(\tau)},\qquad \text{for all } t\in[0,T], \tag{5.6}$$

where $\phi^{i,(\tau)}=(\phi_t^{i,(\tau)})_{0\le t\le T}$ is a $\mathbb{G}^{(\tau)}$-predictable process with $\int_0^T(\phi_t^{i,(\tau)})^2\,\mathrm{d}t<\infty$ P-a.s., for $i=1,2$ and $W^{i,(\tau)}$, $i=1,2$, are the $\mathbb{G}^{(\tau)}$-Brownian motions introduced in (5.1). We can now obtain an explicit representation of a $\mathbb{G}^{(\tau)}$-local martingale $N^{(\tau)}=(N_t^{(\tau)})_{0\le t\le T}$ appearing in Proposition 5.2(b). Combining (5.6) with (5.2)-(5.4) yields, for all $t\in[0,T]$:

$$\begin{aligned} N_t^{(\tau)} \;&=\; \int_0^{t\wedge\tau}\varphi_u^{2,(\tau)}\,\mathrm{d}W_u^{2,(\tau)}+\mathbf{1}_{\{\rho=0\}}\int_\tau^{t\vee\tau}\varphi_u^{1,(\tau)}\,\mathrm{d}W_u^{1,(\tau)} \\ &\quad+\mathbf{1}_{\{\rho\neq 0\}}\left(\int_\tau^{t\vee\tau}\varphi_u^{3,(\tau)}\,\mathrm{d}W_u^{2,(\tau)}-\frac{\sqrt{1-\rho^2}}{\rho}\int_\tau^{t\vee\tau}\varphi_u^{3,(\tau)}\,\mathrm{d}W_u^{1,(\tau)}\right), \end{aligned} \tag{5.7}$$

where $\varphi^{j,(\tau)}=(\varphi_t^{j,(\tau)})_{0\le t\le T}$ is an arbitrary $\mathbb{G}^{(\tau)}$-predictable process satisfying the integrability condition $\int_0^T(\varphi_t^{j,(\tau)})^2\,\mathrm{d}t<\infty$ P-a.s., for $j=1,2,3$. Together with Proposition 5.2, this gives a full characterization of the set of all ELMMs for the model (2.1)-(2.3) in the filtration $\mathbb{G}^{(\tau)}$.

Remark 5.4. A special case of our model, with $\sigma^i(t,X_t)\equiv\sigma$ and $\mu^i(t,X_t)\equiv\mu^i+\sigma^2/2$, for $i=1,2$, $\rho=0$ and τ independent of W^1 and W^2, is described in Cawston and Vostrikova [5, page 13] in the example referred to as a change point Black-Scholes model. Note that the independence of τ and W^i, for $i=1,2$, implies the immersion property between $\mathbb{F}$ and the initially enlarged filtration $\mathbb{G}^{(\tau)}$. In particular, $\theta^{i,(\tau)}\equiv 0$ in (5.1), and thus $W^{i,(\tau)}=W^i$, for $i=1,2$. Proposition 5.2 (b) together with (5.7) implies that the density process Z of any ELMM for the model (2.1)-(2.3) in $\mathbb{G}^{(\tau)}$ admits the representation:

$$Z_t \;=\; Z_0\,\mathcal{E}\left(-\int_0^{\cdot\wedge\tau}\frac{\tilde{\mu}_s^{(\tau)}}{V_s}\,\mathrm{d}W_s^1-\int_\tau^{\cdot\vee\tau}\frac{\tilde{\mu}_s^{(\tau)}}{V_s}\,\mathrm{d}W_s^2+\int_0^{\cdot\wedge\tau}\varphi_s^{2,(\tau)}\,\mathrm{d}W_s^2+\int_\tau^{\cdot\vee\tau}\varphi_s^{1,(\tau)}\,\mathrm{d}W_s^1\right)_t,$$

with $Z_0=f(\tau)$, for some Borel-measurable function $f:\mathbb{R}_+\to\mathbb{R}_+$ such that $E[f(\tau)]=1$ and where the processes $\varphi^{i,(\tau)}=(\varphi_t^{i,(\tau)})_{0\le t\le T}$ are arbitrary $\mathbb{G}^{(\tau)}$-predictable processes satisfying the integrability condition $\int_0^T(\varphi_t^{i,(\tau)})^2\,\mathrm{d}t<\infty$ P-a.s., for $i=1,2$. Note that the representation above contains some additional local martingale terms in comparison to Cawston and Vostrikova [5]. More precisely, these are the last two local martingales on the right-hand side, which are orthogonal to the first two. The representation in Cawston and Vostrikova [5] is thus a special case with $\varphi^{i,(\tau)}\equiv 0$, for $i=1,2$.

5.2. **The initially enlarged filtration $\mathbb{G}^{X,(\tau)}$.** In this section, we study the model (2.1)-(2.3) in the filtration $\mathbb{G}^{X,(\tau)}$. Here we might be concerned with a non-standard initial enlargement since we are enlarging initially the filtration $\mathbb{F}^X$ with respect to which τ can already be

a stopping time. Thus, the classical density hypothesis cannot be imposed[4]. In particular, if Condition III-(a) is satisfied, τ is an $\mathbb{F}^X$-stopping time (Proposition 4.4).

Let us begin by stating the canonical decomposition of X with respect to the filtration $\mathbb{G}^{X,(\tau)}$. It is obtained by projecting the canonical decomposition with respect to $\mathbb{G}^{(\tau)}$ onto $\mathbb{G}^{X,(\tau)}$ and the proof is analogous to that of Lemma 3.6.

Lemma 5.5. *Suppose that Conditions I and IV hold. Then the process X admits the following canonical decomposition with respect to the filtration $\mathbb{G}^{X,(\tau)}$:*

$$X_t = \int_0^t \mu_u^{(\tau)}\,\mathrm{d}u + \int_0^t V_u\,\mathrm{d}B_u^{(\tau)} \qquad \text{for all } t \in [0,T], \tag{5.8}$$

where the $\mathbb{G}^{X,(\tau)}$-predictable process $\mu^{(\tau)} = (\mu_t^{(\tau)})_{0\le t\le T}$ is defined as follows, for all $t \in [0,T]$:

$$\begin{aligned} \mu_t^{(\tau)} \;:=\; & \mathbf{1}_{\{t\le\tau\}}\left(\mu^1(t,X_t) + \sigma^1(t,X_t)\,{}^p\theta_t^{1,(\tau)}\right) \\ & +\mathbf{1}_{\{t>\tau\}}\left(\mu^2(t,X_t) + \sigma^2(t,X_t)\big(\rho\,{}^p\theta_t^{1,(\tau)} + \sqrt{1-\rho^2}\,{}^p\theta_t^{2,(\tau)}\big)\right), \end{aligned} \tag{5.9}$$

with ${}^p\theta^{i,(\tau)}$ denoting the $\mathbb{G}^{X,(\tau)}$-predictable projection of $\theta^{i,(\tau)}$, for $i = 1,2$, and where the process $B^{(\tau)} = (B_t^{(\tau)})_{0\le t\le T}$ is a $\mathbb{G}^{X,(\tau)}$-Brownian motion independent of τ and $V = (V_t)_{0\le t\le T}$ is defined as in (3.4).

Proof. Similarly to the proof of Lemma 3.6, note that since X is a $\mathbb{G}^{(\tau)}$-semimartingale, it is also a $\mathbb{G}^{X,(\tau)}$- semimartingale because of $\mathbb{G}^{X,(\tau)} \subset \mathbb{G}^{(\tau)}$. We denote by $X = A^{(\tau)} + M^{(\tau)}$ the canonical decomposition of X in $\mathbb{G}^{X,(\tau)}$, where $A^{(\tau)}$ is a continuous $\mathbb{G}^{X,(\tau)}$-predictable process of finite variation and $M^{(\tau)} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G}^{X,(\tau)})$ with $A_0^{(\tau)} = M_0^{(\tau)} = 0$. Using the same arguments as in the proof of Lemma 3.6, we obtain $A^{(\tau)} = \int_0^{\cdot} {}^p\tilde{\mu}_t^{(\tau)}\,\mathrm{d}t$, where ${}^p\tilde{\mu}^{(\tau)}$ denotes the $\mathbb{G}^{X,(\tau)}$-predictable projection of $\tilde{\mu}^{(\tau)}$. Now set $\mu^{(\tau)} := {}^p\tilde{\mu}$ and $B^{(\tau)} := \int V^{-1}\,\mathrm{d}M^{(\tau)}$, which is a $\mathbb{G}^{X,(\tau)}$-Brownian motion since $[B^{(\tau)}]_t = t$, for all $t \in [0,T]$. Moreover, $B^{(\tau)}$ is independent of τ by the same arguments as in Remark 5.1-2. □

Let us now check if the model (2.1)-(2.3), considered with respect to the filtration $\mathbb{G}^{X,(\tau)}$, admits arbitrages of the first kind. We denote by ${}^p\theta^{(\tau)}$ the $\mathbb{G}^{X,(\tau)}$-predictable projection of the process $\theta^{(\tau)}$ introduced in (5.5):

$${}^p\theta^{(\tau)} := \mathbf{1}_{[\![0,\tau]\!]}\,{}^p\theta^{1,(\tau)} + \mathbf{1}_{(\!(\tau,T]\!]}\big(\rho\,{}^p\theta^{1,(\tau)} + \sqrt{1-\rho^2}\,{}^p\theta^{2,(\tau)}\big). \tag{5.10}$$

The next proposition characterizes the validity of NA1 with respect to the filtration $\mathbb{G}^{X,(\tau)}$. The proof is analogous to that of Proposition 3.3(a) and hence omitted.

Proposition 5.6. *Suppose that Conditions I and IV hold. Then the model* (2.1)-(2.3) *considered in the filtration $\mathbb{G}^{X,(\tau)}$ satisfies NA1 if and only if $\int_0^T ({}^p\theta_t^{(\tau)})^2\,\mathrm{d}t < \infty$ P-a.s., with the latter condition being equivalent to $\int_0^T (\mu_t^{(\tau)}/V_t)^2\,\mathrm{d}t < \infty$ P-a.s.*

In order to proceed with the study of the properties of the model (2.1)-(2.3) with respect to the filtration $\mathbb{G}^{X,(\tau)}$, we define the $\mathbb{G}^{X,(\tau)}$-adapted continuous process $\widehat{Y}^{(\tau)} = (\widehat{Y}_t^{(\tau)})_{0\le t\le T}$

[4]As we already mentioned in the introduction of Section 5.1, such an initial enlargement can be dealt with using Yor's method; see Yor [42, Chapter 12].

as follows, for all $t \in [0,T]$:

$$\widehat{Y}_t^{(\tau)} := \int_0^t \frac{\mu_u^{(\tau)}}{V_u}\,\mathrm{d}u + B_t^{(\tau)},$$

where $\mu^{(\tau)}$ and $B^{(\tau)}$ are as in Lemma 5.5. Let us denote by $\mathbb{F}^{\widehat{Y}^{(\tau)}} = (\mathcal{F}_t^{\widehat{Y}^{(\tau)}})_{0\le t\le T}$ the right-continuous P-augmented natural filtration of $\widehat{Y}^{(\tau)}$. Below we provide the explicit representation of the filtration $\mathbb{G}^{X,(\tau)}$ as an initial enlargement of the filtration $\mathbb{F}^{\widehat{Y}^{(\tau)}}$ with respect to τ. More precisely, the following result can be proved using the same arguments as in the proof of Lemma 3.9.

Lemma 5.7. *Suppose that Conditions I and IV hold. Then $\mathbb{G}^{X,(\tau)} = \mathbb{G}^{\widehat{Y}^{(\tau)},(\tau)}$, where $\mathbb{G}^{\widehat{Y}^{(\tau)},(\tau)}$ is the initial enlargement of the filtration $\mathbb{F}^{\widehat{Y}^{(\tau)}}$ with respect to τ defined by $\mathcal{G}_t^{\widehat{Y}^{(\tau)},(\tau)} := \bigcap_{s>t}\big(\mathcal{F}_s^{\widehat{Y}^{(\tau)}} \vee \sigma(\tau)\big)$, for all $t \in [0,T]$.*

Making use of Lemma 5.7 we are able to study the filtration $\mathbb{G}^{X,(\tau)}$ using the standard techniques for initial enlargements of filtrations. We begin by proving a predictable representation property for the filtration $\mathbb{G}^{X,(\tau)}$.

Proposition 5.8. *Suppose that Conditions I and IV hold and assume that NA1 holds in the filtration $\mathbb{G}^{X,(\tau)}$. Then every $\mathbb{G}^{X,(\tau)}$-local martingale $L^{(\tau)} = (L_t^{(\tau)})_{0\le t\le T}$ admits a representation of the form:*

$$L_t^{(\tau)} = L_0^{(\tau)} + \int_0^t \varphi_u^{(\tau)}\,\mathrm{d}B_u^{(\tau)}, \qquad \text{for all } t \in [0,T], \tag{5.11}$$

for some $\mathbb{G}^{X,(\tau)}$-predictable process $\varphi^{(\tau)} = (\varphi_t^{(\tau)})_{0\le t\le T}$ with $\int_0^T (\varphi_t^{(\tau)})^2\,\mathrm{d}t < \infty$ P-a.s.

Proof. Since NA1 holds, Proposition 5.6 implies that $\int_0^T (\mu_t^{(\tau)}/V_t)^2\,\mathrm{d}t < \infty$ P-a.s. Thus, the strictly positive $\mathbb{G}^{X,(\tau)}$-local martingale $\widehat{Z}^{(\tau)} := \mathcal{E}\big(-\int(\mu^{(\tau)}/V)\,\mathrm{d}B^{(\tau)}\big)$ is well-defined. We use a localization procedure similar to the one in the proof of Proposition 3.10 to obtain a sequence of uniformly integrable $\mathbb{G}^{X,(\tau)}$-martingales $(\widehat{Z}^{(\tau),\tau_n})_{n\in\mathbb{N}}$, where (τ_n) is a sequence of $\mathbb{G}^{X,(\tau)}$-stopping times. For every $n \in \mathbb{N}$, we define the filtrations $\mathbb{G}^{X,(\tau),n} := (\mathcal{G}_{t\wedge\tau_n}^{X,(\tau)})_{0\le t\le T}$, $\mathbb{F}^{\widehat{Y}^{(\tau)},n} := (\mathcal{F}_{t\wedge\tau_n}^{\widehat{Y}^{(\tau)}})_{0\le t\le T}$ and $\mathbb{G}^{\widehat{Y}^{(\tau)},(\tau),n} := (\mathcal{G}_{t\wedge\tau_n}^{\widehat{Y}^{(\tau)},(\tau)})_{0\le t\le T}$ and the probability measure Q^n on $\mathcal{G}_{T\wedge\tau_n}^{X,(\tau)}$ by $\frac{\mathrm{d}Q^n}{\mathrm{d}P} := \widehat{Z}_{T\wedge\tau_n}^{(\tau)}$. By Girsanov's theorem the stopped process $\widehat{Y}^{(\tau),\tau_n}$

$$\widehat{Y}_t^{(\tau),\tau_n} = B_t^{(\tau),\tau_n} + \int_0^{t\wedge\tau_n} \frac{\mu_u^{(\tau)}}{V_u}\,\mathrm{d}u = B_t^{(\tau),\tau_n} - \int_0^{t\wedge\tau_n} \frac{1}{\widehat{Z}_u^{(\tau)}}\,\mathrm{d}[\widehat{Z}^{(\tau)}, B^{(\tau)}]_u^{\tau_n}$$

is a $(Q^n, \mathbb{G}^{X,(\tau),n})$-local martingale and also a stopped $(Q^n, \mathbb{F}^{\widehat{Y}^{(\tau)},n})$-Brownian motion. Furthermore, since $\widehat{Y}^{(\tau),\tau_n}$ is also a stopped $(Q^n, \mathbb{G}^{X,(\tau),n})$-Brownian motion, it follows that τ and $\mathbb{F}^{\widehat{Y}^{(\tau)},n}$ are independent. Hence, the filtration $\mathbb{F}^{\widehat{Y}^{(\tau)},n}$ is immersed into the initial enlargement $\mathbb{G}^{\widehat{Y}^{(\tau)},(\tau),n}$ and the density hypothesis is trivially satisfied. The predictable representation property in $\mathbb{G}^{\widehat{Y}^{(\tau)},(\tau),n}$ is thus straightforward: every $(Q^n, \mathbb{G}^{\widehat{Y}^{(\tau)},(\tau),n})$-local martingale $\hat{L} = (\hat{L}_t)_{0\le t\le T}$ can be represented as

$$\hat{L}_t = \hat{L}_0 + \int_0^t \varphi_u^{(\tau)}\,\mathrm{d}\widehat{Y}_u^{(\tau),\tau_n}, \qquad \text{for all } t \in [0,T], \tag{5.12}$$

where $\varphi^{(\tau)}$ is a $\mathbb{G}^{\widehat{Y}^{(\tau)},(\tau),n}$-predictable process such that $\int_0^T(\varphi_t^{(\tau)})^2\,\mathrm{d}t<\infty$ P-a.s. (see Proposition 5.3(i) in Callegaro et al. [4]). Now let $L^{(\tau)}=(L_t^{(\tau)})_{0\leq t\leq T}$ be a $(P,\mathbb{G}^{X,(\tau)})$-local martingale. Resorting to the sequence of stopped processes $(L^{(\tau),\tau_n})$ and repeating similar steps as in the last part of the proof of Proposition 3.10, we complete the proof. □

The previous proposition allows to characterize the validity of NFLVR in $\mathbb{G}^{X,(\tau)}$.

Proposition 5.9. *Suppose that Conditions I and IV hold. Then NFLVR holds in the filtration* $\mathbb{G}^{X,(\tau)}$ *if and only if NA1 holds and* $E[\mathcal{E}(-\int(\mu^{(\tau)}/V)\,\mathrm{d}B^{(\tau)})_T]=1$.

Proof. As in the proof of Proposition 5.2(b), NFLVR holds in $\mathbb{G}^{X,(\tau)}$ if and only if there exists a martingale deflator Z in $\mathbb{G}^{X,(\tau)}$such that $E[Z_T]=1$. The claim follows from Lemma 5.5 and Proposition 5.8 combined with Fontana and Runggaldier [14, Lemma 4.3.15]. □

As expected, the absence of arbitrage in the larger filtration $\mathbb{G}^{(\tau)}$ implies the absence of arbitrage in the smaller filtration $\mathbb{G}^{X,(\tau)}$. More precisely, along the same lines of Proposition 3.12 we obtain:

Proposition 5.10. *Suppose that Conditions I and IV hold. If the model* (2.1)-(2.3) *satisfies NA1 (resp. NFLVR) in the filtration* $\mathbb{G}^{(\tau)}$, *then NA1 (resp. NFLVR) holds in the filtration* $\mathbb{G}^{X,(\tau)}$ *as well.*

As a consequence of Proposition 5.8, the financial market $(S,\mathbb{G}^{X,(\tau)})$ is complete (up to a random initial endowment). More precisely, we have:

Corollary 5.11. *Under the same assumptions as in Proposition 5.8, any bounded* $\mathcal{G}_T^{X,(\tau)}$-*measurable non-negative random variable* H *admits the representation* $H=f^H(\tau)+\int_0^T h_t^H\,\mathrm{d}S_t$ P*-a.s., for some Borel-measurable function* $f^H:\mathbb{R}_+\to\mathbb{R}_+$ *and for* $h^H\in L(S,\mathbb{G}^{X,(\tau)})$.

Proof. The claim follows from equation (5.8) and Proposition 5.8 by the same arguments as in the proof of Corollary 4.7, with the only difference that the initial σ-field is no longer trivial, since $\mathcal{G}_0^{X,(\tau)}=\sigma(\tau)$. □

We conclude this section by stating explicitly the connection between the Brownian motion B in the progressively enlarged filtration $\mathbb{G}^X$ (see Section 3.2) and the Brownian motion $B^{(\tau)}$ in the initially enlarged filtration $\mathbb{G}^{X,(\tau)}$.

Remark 5.12 (*Connection between the Brownian motions* B *and* $B^{(\tau)}$)**.** As shown in the proof of Lemma 3.6, the $\mathbb{G}^X$-Brownian motion B is given by $B=\int V^{-1}\,\mathrm{d}M$, where M is the local martingale part in the $\mathbb{G}^X$-canonical decomposition of X. On the other hand, the $\mathbb{G}^{X,(\tau)}$-Brownian motion $B^{(\tau)}$ is defined as $B^{(\tau)}=\int V^{-1}\,\mathrm{d}M^{(\tau)}$, where $M^{(\tau)}$ is the local martingale in the $\mathbb{G}^{X,(\tau)}$-canonical decomposition of X, as in the proof of Lemma 5.5. Now combining equations (3.9) and (5.8) we obtain

$$B_t=B_t^{(\tau)}+\int_0^t\frac{1}{V_s}\left({}^p\tilde{\mu}_s^{(\tau)}-{}^p\tilde{\mu}_s\right)\mathrm{d}s,\qquad\text{for all } t\in[0,T],\tag{5.13}$$

where ${}^p\tilde{\mu}^{(\tau)}$ is the $\mathbb{G}^{X,(\tau)}$-predictable projection of the process $\tilde{\mu}^{(\tau)}$ defined in (5.3) and ${}^p\tilde{\mu}_s$ is the $\mathbb{G}^X$-predictable projection of the process $\tilde{\mu}$ defined in (3.3). Note that the process $\frac{1}{V}\left({}^p\tilde{\mu}^{(\tau)}-{}^p\tilde{\mu}\right)$ is $\mathbb{G}^{X,(\tau)}$-predictable since V and ${}^p\tilde{\mu}^{(\tau)}$ are $\mathbb{G}^{X,(\tau)}$-predictable and ${}^p\tilde{\mu}$ is $\mathbb{G}^X$-predictable (recall that $\mathbb{G}^X\subset\mathbb{G}^{X,(\tau)}$). If Condition III-(a) holds, $\mathbb{F}^X=\mathbb{G}^X$ and thus, B is

also an $\mathbb{F}^X$-Brownian motion and then (5.13) gives the connection between the $\mathbb{F}^X$-Brownian motion B and $\mathbb{G}^{X,(\tau)}$-Brownian motion $B^{(\tau)}$. Note that equation (5.13) gives the canonical decomposition of the $\mathbb{G}^X$-Brownian motion in the enlarged filtration $\mathbb{G}^{X,(\tau)}$ and, furthermore, gives a precise link between the canonical decompositions of the process X in the two filtrations $\mathbb{G}^X$ and $\mathbb{G}^{X,(\tau)}$ (compare with Lemma 3.6 and Lemma 5.5).

6. Conclusion

In this paper we have studied a class of asset price models with a change point, imposing only minimal assumptions on the random time and on the driving Brownian motions. We characterize the model by its properties of martingale representation, completeness or incompleteness and two notions of arbitrage. The analysis of the model is undertaken in all the filtrations that naturally arise in this framework.

The model can be generalized to incorporate multiple change points by enlargement of filtrations with a sequence of random times and to incorporate discontinuous semimartingales as the driving processes by theories and techniques for jump processes. If a martingale representation property holds in a filtration generated by the two underlying semimartingales (this is true e.g. for Lévy processes or marked point processes), then martingale representation theorems can also be obtained for various filtrations related to the model. Similarly, our results can also be generalized to the case where the drift term μ and the diffusion term σ in (2.2) are $\mathbb{F}$-progressively measurable processes instead of functions of (t, X_t). The extension of other results from this paper, in particular the characterization of the no-arbitrage properties, crucially depends on the specific class of semimartingales used as driving processes.

Besides the obvious application to the modeling of financial asset prices, the results obtained in the present paper can also be useful for the development of random switching models for credit risk, interest rate modeling and energy markets. Indeed, in those contexts one naturally observes the presence of sudden changes in the dynamics of prices and other financial variables at random time points.

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